Comments on the quiz! (July 9, 2008)

Here are some items you might think over — several students made the same kind of errors.

1 a det A makes no sense if A is 5×3. But if A has rank 3, then $A^T A$ will be nonsingular. This is the basic principle behind the notion of projection on subspaces (with a complicated formula).

d) The rank of A has nothing to do with this issue. But rank still does come in. The point is we look at $\det(A - \lambda I)$, and this is a polynomial of degree n in λ. So if $\lambda = \lambda_0$ is a solution, we then do consider the rank of $A - \lambda_0 I$. Its determinant is zero, and so there is a nonzero vector $\underline{x}$ with $(A - \lambda_0 I)\underline{x} = 0$! $A\underline{x} = \lambda_0 \underline{x}$.

We are not allowed to take $\underline{x} = 0$! ! The zero vector is never an eigenvector.

3. The first column, $q_1 = \binom{\cos\theta}{\sin\theta}$ is already of length 1. So the second column of $Q = (q_1\, q_2)$ is obtained by subtracting the projection of $\binom{\sin\theta}{0}$ onto q_1 from q_1, then making it length one. So

$$q_2{}' = \binom{\sin\theta}{0} - \underbrace{(\cos\theta\sin\theta)\binom{\sin\theta}{0}}_{\text{scalar}}\binom{\cos\theta}{\sin\theta}.$$ We get $q_2{}' = \sin^2\theta\binom{\sin\theta}{-\cos\theta}$,

$q_2 = \binom{\sin\theta}{-\cos\theta}$.

Key fact if λ is an eigenvalue and A is an $n \times n$ matrix, then there is always a nontrivial solution to $Ax = \lambda x$ (if we allow complex entries).

A famous unsolved question asks the same thing in infinite dimensions [I am a little vague here]. If A is a (continuous) linear transformation on a vector space X [of any dimension], is there a subspace of X (other than all of X or just the subspace consisting of the zero vector) which A leaves invariant?