§5.5 Complex matrices. New vocabulary.

Review complex addition, multiplication, conjugation.

$z = a+ib$; $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$, $|z|^2 = z\bar{z}$ (something new)

Dictionary.

$$\|x\|^2 = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 = x_1\bar{x}_1 + x_2\bar{x}_2 + \cdots + x_n\bar{x}_n \overset{!}{=} x^H x$$

where A^H is the conjugate transpose of A. (Otherwise $x^T x$ might be negative $\binom{i}{i}(i,i) = -2$!

(pp 285, p289)

$\boxed{A = P\Lambda Q^T = \sum \lambda_i x_i x_i^H}$

Hermetian:	$A^H = A$	(Symmetric)
Unitary	$U^H U = I$	(orthogonal)
Orthogonal	$x^H y = 0$	

$$(Ax)^H y = x^H (A^H y) \quad [= (x^H A^H) y]$$

$$(AB)^H = B^H A^H$$

Note that $y^T x \neq x^T y$ (for this part, all vectors are complex!).

Goal: Hermetian matrices can be diagonalized by an ON set of basis vectors. Let A be Hermetian

① Eigenvalues of A are real

(A) $x^H A x$ is always real if A is Hermetial

(real symm ⊂ Hermetian)

(Proof. $x^H A x$ is a scalar, so its complex conjugate is $\overline{(x^H A x)} = (x^H A x)^H = x^H A^H x^{HH} = x^H A x$. So $\overline{x^H A x} = x^H A x$, and is real.

(p 284, book does 2X2 case directly)

② $Ax = \lambda x$, A Hermetian $\Rightarrow \lambda$ real. Try

real $= x^H A x = x^H \lambda x = \lambda x^H x$, but $x^H x$ is real.

③ Eigenvalues of Hermetian matrix w. different eigenvalues are $\perp$.

Indeed, let $Ax = \lambda_1 x$, $Ay = \lambda_2 y$, $(A = A^H)$

$$(\lambda_1 x^H y) = (y, \lambda_1 x) \equiv (\lambda_1 x)^H y = (Ax)^H y \overset{!}{=} x^H (Ay) = \lambda_2 (x^H y).$$

But $\lambda_1 \neq \lambda_2$, so $x^H y \left(= (x,y)\right) = 0$

Thus for a Hermitian matrix, the eigenspaces are orthogonal, So we can make an orthogonal matrix of eigenvectors, Q, (This is even true if eigenvalues are repeated, but we'll not say too much about that.

Analogue of Q: unitary matrix U: $U^H U = I = U U^H$.

<u>Note</u>

$$\|Ux\|^2 = \|x\|^2,$$

if $Ux = \lambda x$, then $|\lambda| = 1$; (they need not be real!)

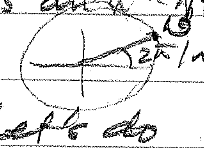
and $(x,y) = x^H y = 0$ if $Ux = \lambda_1 x$, $Uy = \lambda_2 y$, $\lambda_2 \neq \lambda_1$

(to see this, look at $x^H y = (Ux)^H (Uy)$)

Important example from FFT!

$$U = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & & \\ 1 & \omega^{n-1} & (\omega^{n-1})^2 & \cdots & & \end{bmatrix}$$ where ω is an n^{th} root of 1!

$$\omega = e^{2\pi i/n}$$



To see U is unitary we need geometric series. Let's do row 2 of U $\times$ col 3 of U^H:

$$(1, \omega, \omega^2 \ldots \omega^{n-1}) \cdot (1\, \bar{\omega}^2, \bar{\omega}^{2\cdot 2} \ldots \bar{\omega}^{(n-1)2}) = 1 + \omega^{-1} + \omega^{-2} + \cdots + \omega^{-(n-1)}$$

$$= \frac{1 - \omega^{-n}}{1 - \omega^{-1}} = 0 \quad (\text{denom} \neq 0)$$

for row 2 $\times$ row 2 we get n^2, but we take sq roots and divide by $\sqrt{n}$.

P 289

5(a) $x = re^{i\theta}$. But

$e^{i\theta} = \cos\theta + i\sin\theta$ so $\overline{e^{i\theta}} = \cos\theta - i\sin\theta = e^{-i\theta}$

So $x^2 = r^2 e^{2i\theta}$, $x^{-1} = r^{-1} e^{-i\theta}$, $\bar{x} = re^{-i\theta}$

If $x^{-1} = \bar{x}$, then $\frac{1}{r} e^{i\theta} = re^{-i\theta}$ so $r = 1$ and $|x| = 1$.

9(a) $A^H = \overline{(A^T)}$. $\det A^T = \det A$.
And if we conjugate the entries in A (or A^T)
we get the conjugate ($\overline{xy} = \bar{x}\bar{y}$ ...) So
$\det A^H = \overline{\det A}$.
So if A is Hermetian, $\det A^H = \det A = \overline{\det A}$ so
$\det A$ is real.

P 289 11 b. Q is Hermetial, $\det(Q - \lambda I) = \lambda^2 - 1$,
so $\lambda = \pm 1$, Eigenvectors $\lambda = 1$: $\begin{pmatrix}1\\1\end{pmatrix}$, $\lambda = -1$: $\begin{pmatrix}1\\-1\end{pmatrix}$.
(and they are $\perp$) So we should have

$$P = 1(1,1)\begin{pmatrix}1\\1\end{pmatrix} + -1(1,-1)\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1&1\\1&1\end{pmatrix} - \begin{pmatrix}1&-1\\-1&1\end{pmatrix} =$$
$$\begin{pmatrix}1&1\\1&1\end{pmatrix} + \begin{pmatrix}-1&1\\1&-1\end{pmatrix} = \begin{pmatrix}0&2\\2&0\end{pmatrix}$$

(I didn't take unit vectors!)

13 (part) (a) they are orthogonal

(b) A has rank 2, since 0 is an eigenvalue of mult. 1.
Thus nullspace has dim 1, row space has dim 1. So columnspace has dim 2.

Left nullspace? $z^H A = 0 \Rightarrow A^H z = 0 \Rightarrow A^H z = 0$
since A is Hermetian.