7/14 FFT (Cooley–Tukey 1965; Math. Comp.)

Matrix $F = F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{bmatrix}$. We take $n = 2^m$ so

n is huge when m is not very! $m \sim \log n$.

Goal: solve the system

(1) $$F c = y,$$

where this means at $e^{(2\pi i/n)k} c_k = y_k$ – the Fourier series

$$c_0 + c_1 e^{ix} + c_2 e^{2ix} + \cdots + c_{n-1} e^{(n-1)ix}$$

has appropriate values when $x = e^{2\pi i/n}$ (and powers of x).

And (1) is not hard to solve, because F^{-1} is so easy to see:

$$F^{-1} = \frac{1}{n} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-(n-1)} & \cdots & \omega^{-(n-1)^2} \end{bmatrix} = \frac{\overline{F}}{n}!$$

F^{-1} is about as simple as F. Of course this means that computing Fc, $F^{-1}y$ should be quick.

Key Fact: if $m = \frac{1}{2}n$, then $\omega_n^2 = \omega_m$

This makes the computation in (1) (or taking its inverse) simpler computationally than one would expect (2^n is huge),

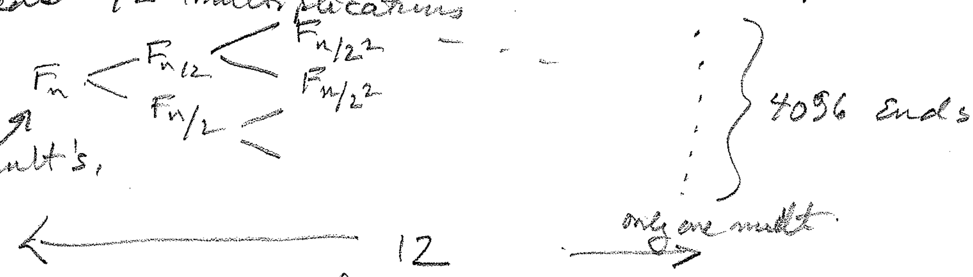
To compute $y = F_n c$, we let $m = \frac{1}{2}n$, and write c as two pieces:

$$c' = (c_0, c_2, \ldots, c_{n-2}); \quad c'' = (c_1, c_3, \ldots, c_{n-1}),$$

leading to $y' = F_m c'$, $y'' = F_m c''$ in place of $y = F_n c$.

Note: $m = \frac{1}{2}n$, and we have two equations rather than one – think of $m = 12$ so that $2^m = 4096$, still rather small!

We then can go to $F_{\frac{1}{2}m}$, $F_{\frac{1}{2^2}m}$, … We get to $m = 1$ from $n = 2^{12}$ by this procedure in 12 steps. Each step as n decreases needs $n/2$ multiplications



(book says $\frac{1}{2} \cdot 12$)

7/14 p2

Flow chart (p 195) $n=4$ to $m=\frac{1}{2}n=2$

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \longrightarrow \begin{bmatrix} c_0 \\ c_2 \\ c_1 \\ c_3 \end{bmatrix} \longrightarrow \begin{bmatrix} F_2 c' \\ \\ F_2 c'' \end{bmatrix} \longrightarrow \begin{bmatrix} \\ y \\ \\ \end{bmatrix}$$

Do via matrices, now from right to left. We apply to $\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$:

$(\star)$ (1st entry always 1)

$$\begin{bmatrix} 1 & & 1 & \\ & 1 & & \omega_4 \\ 1 & & -1 & \\ & 1 & & -\omega_4 \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}$$

mult by ω_n^j $\qquad$ F_2 $\qquad$ $c \to c', c''$

Analytically:

(j is fixed) $y_j = \sum_0^{n-1} \omega_n^{jk} c_k \iff y_j = \sum \omega_n^{2kj} c_{2k} + \sum \omega_n^{(2k+1)j} c_{2k+1}$

even $\qquad$ odd

$$= \sum_0^{m-1} \omega_m^{kj} c_k' + \omega_n^j \sum \omega_m^{kj} c_k''$$

$$= y_j' + \omega_n^j y_j''$$

Now we consider the cases $0 \le j \le m-1$ and $m+j$, $0 \le j \le m-1$. In the first case $\omega_n^j = \omega_n^j$, nothing new. But in the second, $\omega_n^{j+m} = -\omega_n^j$ since $\omega_n^m = -1$! So we have the system

$$y_j = y_j' + \omega_n^j y_j'' \qquad 0 \le j \le m-1$$

$$y_{j+m} = y_j' - \omega_n^j y_j'' \qquad 0 \le j \le m-1$$

The displayed matrix $(\star)$ is simply F_4 (check!)

p 196 # 5 $e^{ix} = -1 = e^{i\pi}$, Since $e^{2\pi i} = 1$, this means

$x = \pi + 2\pi n$ where n is an integer. So there are infinitely many solutions;

$$x = \pi(1+2n), \quad n = 0, \pm 1, \pm 2, \dots$$

18 Let's look at the columns of the left side.

1st $P \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ where $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

So $P \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\lambda_0 = 1$. Next,

$P \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix} = \begin{bmatrix} i \\ i^2 \\ i^3 \\ 1 \end{bmatrix} = i \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix}$ so $\lambda_1 = i$, etc.

Appendix B – Jordan form.

"Bottom line". Let $\lambda = \lambda_0$ be a root of the characteristic polynomial of multiplicity k. We'd like to see that this gives, with a good choice of basis, a matrix of the form

$$\begin{bmatrix} \lambda_0 & & 0 \\ & \lambda_0 & \\ 0 & & \ddots \; \lambda_0 \end{bmatrix} \quad k \times k$$

but this is not possible. The best we can do is the Jordan form! We get upper Δlar, and 0's 2 diagonals and more above the main diagonal, but that's about it.

Ex $AM = MJ$ $\qquad M = (x_1 x_2 \dots x_n)$

$$\begin{bmatrix} 2 & 1 & 0 & 0 & \cdots & \\ & 2 & 1 & & & \\ & & 2 & 0 & 0 & \\ & 0 & & 2 & 1 & \\ & & & & 2 & \\ & & & & & 2 \end{bmatrix}$$

Top left $\quad Ax_1 = 2x_1$

$Ax_2 = 2x_2 + x_1$

$Ax_3 = 2x_3 + x_2$

Then $\quad Ax_4 = 2x_4$

$Ax_5 = 2x_5 + x_4$

$Ax_6 = 2x_6$

(3 strings of vectors).

P427 1.a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ Eigenvalues are $\lambda = 0, 2$, so

Jordan form is $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

(b) Char. polynomial is $-\lambda^3$ so $\lambda = 0$ is only root. The matrix has rank 1 and so there are two l.i. eigenvectors. J-form is

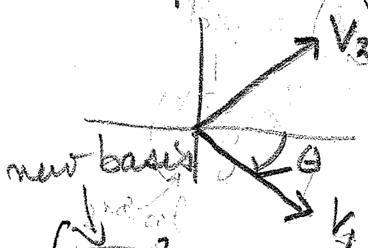
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \left(B = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

§5.6 Similarity, Relation of matrices

Book notation: $[I]_{V \text{ to } v}$ means:

Write vectors of V in terms of v, and put the entries in the columns of $[I]_{V \text{ to } v}$. (but V, v are coordinates!)

Book example: Projection on ray $\arg z = \theta$.



$V_2 \longrightarrow 0$

$$M = [I]_{V \text{ to } v}: \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad [T]_V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$[I]_{v \text{ to } V} = [I]_{V \text{ to } v}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Then $AM = MT$, $A = MTM^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

in standard basis