If $B = M^{-1}AM$ then A and B have same eigenvalues. If x is an eigenvector of A, then $M^{-1}x$ is the corresponding eigenvalue of B. (p 296)

We like B (or T) to be relative to eigenvectors as basis,

(check with f $M = \begin{pmatrix}\cos x & -\sin x \\ \sin x & \cos x\end{pmatrix}$)

Spectral Theorem

A real symm. $\Rightarrow A = Q\Lambda Q^T$ Λ diag, Q orth

A complex hermetian $\Rightarrow A = U\Lambda U^T$ Λ diag, U unitary

In general (p 296) we have:

There is a unitary M so that $U^{-1}AU = T$ is triangular.

Remark Why are symmetric matrices better?

Any matrix may be approximated by one with distinct roots, so diagonalizable. Book notes (p297) that in symmetric case, the corresponding Us are ON columns, and so in the limit we expect an ON basis of eigenvectors

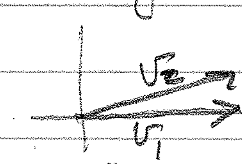
But now let's compare the "close matrices" A (not diagonalizable)

$A: \begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}$

0 only evalue, evector $\begin{pmatrix}1 \\ 0\end{pmatrix}$

$B: \begin{pmatrix}0 & 1 \\ 0 & \varepsilon\end{pmatrix}$ 0, ε evalues

evector for 0: $\begin{pmatrix}1 \\ 0\end{pmatrix}$ evector for ε: $\begin{pmatrix}1 \\ \varepsilon\end{pmatrix}$



What happens in limit

In limit $v_1 \to 0$

$v_2 \to v_1 \to 0$

$\begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}$ $v_1 = \begin{pmatrix}1 \\ 0\end{pmatrix}$ $v_2 = \begin{pmatrix}0 \\ 1\end{pmatrix}$

How to get Δlar form (p 296).

Given A, find one (unit) e-vector x_1 (this is guaranteed)

Put x_1 in first column of U and check

$U_1^{-1} A U_1 = \Lambda_1$; $A \underset{U_1 \uparrow}{(x_1 * * \cdots)} = \underset{U_1}{(x_1 * \cdots)} \underset{\Lambda_1}{\begin{pmatrix} \lambda_1 & \\ 0 & \\ 0 & \\ \vdots & \end{pmatrix}}$ U_1 is 1st approximation, U_1 is unitary

Now take U_2 as

$$U_2 = \left(\begin{array}{c|c} 1 & 0 \; - - - \\ 0 & \\ 0 & M_2 \\ 0 & \end{array}\right), \quad M_2 \text{ to be determined}$$

We will make U_2 unitary, but a key observation is that we won't touch the first row or column — so $U_1 U_2$ has the same first column as U_1!! we haven't changed x_1!

So if we work in the $(n-1)\times(n-1)$ submatrix of A, we get an evalue λ_2, evector x_2, which we make an n-vector x_2 with first coordinate zero;

$$U_2 = \left(\begin{array}{c|c|c} 1 & 0 & 0 \; 0 \; - - \\ 0 & & \\ 0 & x_2 & \\ 0 & & \\ \vdots & & \end{array}\right) \text{ — fill out to make unitary}$$

Then $U_2^{-1} A_1 U_2 = \underset{\Lambda_2}{\begin{bmatrix} \lambda_1 & * \\ 0 & \lambda_2 \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix}}$; $A_1 = U_1$

← This becomes T in the end

Def A is normal if $AA^{*} = A^{*}A$

Check $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not normal

No accident! Spectral theorem holds iff A is normal

(half of this is easy to see).

Jordan form If A has s λ_i and eigenvectors

then to each λ_i is a block

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

Recall the three 3×3 matrices I gave at the beginning of this chapter, all of whom had $-\lambda^3$ as the characteristic polynomial

Interpretation

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$AM = MJ \qquad MJ = (x_1 x_2 x_3)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This means

$$A(x_1\ x_2\ x_3) = \begin{pmatrix} x_1^1 & x_2^1 & x_3^1 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1^3 & x_2^3 & x_3^3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(Ax_1\ Ax_2\ Ax_3) = (0\ x_1\ x_2)$$

so $Ax_1 = 0$, $Ax_3 = x_2$ $Ax_2 = x_1$

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In general,

$$(J_i)^k = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} \\ 0 & \lambda^k & \binom{k}{1}\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}$$

$$\left(= \left(\lambda I + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)^k \right)$$

I like book account on p 301

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}' = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \frac{du}{dt} = Ju \rightarrow$$

Solution is $u(t) = e^{Jt}u(0)$. So to see third column of e^{Jt}, take $u(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Then we have

$$u_3' = \lambda u_3 \implies u_3 = e^{\lambda t} \quad (\text{since } u(0)=1)$$

Then $u_2' = \lambda u_2 + u_3 = \lambda u_2 + e^{\lambda t} \implies u_2 = te^{\lambda t}$

and similarly $u_1 = \frac{1}{2}t^2 e^{\lambda t}$

Some hw

p 196 #1 $F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \quad F^4 = \begin{pmatrix} 16 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix}$

p 196 #11 $y = F_4 c$ We compute directly first:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

Follow steps on p 195

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \text{even} \\ \text{odd} \end{matrix} \rightarrow \begin{matrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{matrix} y_0' \\ y_1' \end{matrix} \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{matrix} y_0'' \\ y_1'' \end{matrix} \end{matrix}$$ then back

as on p 194 bottom (or 3W)

$$y_0 = y_0' + (i)^0 y_0'' = 2$$
$$y_1 = y_1' + i \cdot 0 = 0$$
$$y_2 = y_{0+2} = y_0' - (i)^0 y_0'' = 2$$
$$y_3 = 0$$

p 196 #18 If the columns are e-vectors of P, then PF has as columns $(\lambda_0 f_0 \ \lambda_1 f_1 \ \lambda_2 f_2 \ \lambda_3 f_3)$. So we look at this as see that $\lambda_0 = 1$. For the second col of PF we get (left side) $\begin{pmatrix} i_2 \\ i_3 \\ i_4 \end{pmatrix}$ so $\lambda_1 = i$. Similarly $\lambda_2 = i^2 = -1$, $\lambda_3 = i^3 = -i$.

cols $(f_0 \ f_1 \ f_2 \ f_3)$

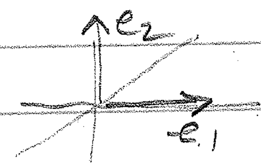
p 302

#2 Anything of the form $M\begin{bmatrix}1&0\\0&-1\end{bmatrix}M^{-1}$.

If $M = \begin{pmatrix}0&1\\1&0\end{pmatrix} = M^{-1}$ we get $\begin{pmatrix}-1&0\\0&1\end{pmatrix}$

Then try $M = \begin{pmatrix}1&1\\0&2\end{pmatrix}$ so $M^{-1} = \frac{1}{2}\begin{pmatrix}2&-1\\0&1\end{pmatrix}$ and

$$M\begin{pmatrix}1&0\\0&-1\end{pmatrix}M^{-1} = \begin{pmatrix}1&-1\\0&-1\end{pmatrix} \quad \text{(I think!)}$$

#11 

$Te_1 = e_2$

$Te_2 = e_1$ so Matrix of T is

$A = \begin{pmatrix}0&1\\1&0\end{pmatrix}$

With respect to $\overset{V_1}{(1,1)}$ and $\overset{V_2}{(1,-1)}$ we have $TV_1 = V_1$, $TV_2 = -V_2$ so the matrix with respect to the V's is

$B, \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

Using notation from p 295, we get

$$M = [I]_{\underset{\text{my word}}{V \text{ from } v}} = \begin{pmatrix}1&1\\1&-1\end{pmatrix};\ M^{-1} = -\frac{1}{2}\begin{pmatrix}-1&-1\\-1&1\end{pmatrix} = \begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&-\frac{1}{2}\end{pmatrix}$$

So $A = MBM^{-1}$: $\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1/2&1/2\\1/2&-1/2\end{pmatrix}$

Ignore p 305 38

Hint for 41c (think of off-diagonal entries)