Ch 6 – Positive Definiteness

Calc in 1 variable: extremum $f'=0$, $f''\neq 0$ ($f''=0$, test fails – need to look at higher-order terms).

Look at (0,0) and use quadratic formula:

f_{xy} $\quad f = ax^2+2bxy+cy^2 \qquad ac\neq 0$

$= a(x+\frac{b}{a}y)^2 + (c-\frac{b^2}{a})y^2$.

~~Of course~~ $f(0,0)=f_x(0,0)=f_y(0,0)=0$.

Clearly: if $a>0$ & $c-b^2/a>0$ then local min

if <0 $\quad <0$ max

(Saddle point) if signs differ, then can be + or −

Def: <u>Positive definite</u> means >0 unless x, y both zero.

So for P.D: $a>0$, $ac>b^2$; neg def: $a<0$ but $ac>b^2$

Write as symmetric matrix $\begin{pmatrix} a & b \\ b & c\end{pmatrix}$ — as in $(x\ y)\begin{pmatrix} a & b \\ b & c\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}$ symmetric

In $\mathbb{R}^n$, write as x^TAx where A is <u>symmetric</u>

This is called a quadratic form. "second deriv. matx"

¿How do formulas change if at (x_0, y_0) or $(x_0^1, \dots, x_0^n)$?

<u>P 316 4b</u> $F=(x^2-2x)\cos y$ at $(1,\pi)$. We should check that $F_x = F_y = 0$ at $(1,\pi)$

So $A = \begin{bmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy}\end{bmatrix} = \begin{bmatrix} 2\cos y & (2-2x)\sin y \\ (2-2x)\sin y & -(x^2-2x)\cos y\end{bmatrix}$;

at $(1,\pi)$ this is $\begin{bmatrix} -2 & 0 \\ 0 & -1\end{bmatrix}$ – local max

Note on Ex 5 p 314: Write as

$2(x_1-\frac{1}{2}x_2)^2 - \frac{1}{2}x_2^2 + 2x_2^2 + 2(x_3-\frac{1}{2}x_2)^2 - \frac{1}{2}x_2^2$ so P.D.

§6.2 Th. The following are equivalent: (so A is P.D.) A is real symmetric

(1) $x^TAx > 0$ if $x \neq 0$ (this is the def of P.D.)

(2) all $\lambda_i > 0$

(3) all "principle minors" have positive determinant

(4) all pivots $d_k > 0$ (; no row exchanges!)

Note a clue from §6.1: we need $a > 0$ (pivot) AND $ac - \frac{b^2}{a} > 0$ (pivot)

Proof (2) → (1) (spectral theorem)

(1) → (3) Look at (1), but for each $1 \le k \le n$, let x have 0's after the k^{th} entry. Then

$$x^TAx = (x_k \ 0)\, A \begin{pmatrix} x_k \\ 0 \end{pmatrix} = x_k^T A_k x_k > 0,$$

so A_k is p.d. and must have pos e'values, pos det!

(3) ⇒ (4) See formula (5) p 224; d_k is ratio of positive numbers.

(4) → (1) We follow Ex 1 p 319 (which is a general argument but easier to follow in an example).

Use elimination to write $A = LDL^T$, L's having 1s on diagonal – it is really completing the square.

We recall (p 51) if A is symmetric, then

$$A = LDL^T$$

We follow the model on p 51 for the example here:

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Get to upper △lar. So mult on left by (in order) $E_{21}(\frac{1}{2})$ and $E_{32}(\frac{2}{3})$,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\underbrace{\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}}_{A} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{pmatrix}$$

Upper △lar!