Name: Solutions **Math 511** **July 21, 2008**

(*Work will be graded on the basis of clarity as well as accuracy.*)

(20) 1. True-False. Write T or F and give a reason in each case If the answer is F, it would be convincing to give an example.

(a) If M is an $n \times n$ matrix with n distinct eigenvalues, then M is similar to a diagonal matrix. T – n lin ind e-vectors

* (b) $e^{A+B} = e^A e^B$ (here A and B are square matrices). F Need $AB = BA$ for example $\frac{1}{2}(A+B)^2 = \frac{1}{2}(A^2+AB+BA+B^2)$ But in $e^A e^B$ we have $\frac{1}{2}(A^2+B^2)+AB$

(c) If A has n distinct eigenvalues, then it is invertible. Not if 0 is an eigenvalue

(d) If a triangular matrix M is similar to a diagonal matrix, then it is already diagonal.

False! try $\begin{pmatrix}1&2\\0&2\end{pmatrix}$ – it is similar to $\begin{pmatrix}1&0\\0&2\end{pmatrix}$

But $\begin{pmatrix}2&2\\0&2\end{pmatrix}$ is NOT similar to $\begin{pmatrix}2&0\\0&2\end{pmatrix}$

(15) 2. let F be a singular $n \times n$ matrix. Which of the following is *not* necessarily true for F?

(a) det $e^F \neq 0$ T

(b) det $(F + I) = 1+$ det F NO (try example)

(c) F is similar to an upper triangular matrix

(d) det $(Fe^F) =$ det $F/$ det(e^{-F})

(e) rank $(FF^T) < n$

T

* Be careful – many of you claimed that if $A = \begin{pmatrix}0&0\\0&0\end{pmatrix}$ then $e^A = 0$. But $e^A \neq 0$, and for this A, $e^A = e\begin{pmatrix}1&0\\0&1\end{pmatrix}$

(20) 3. Let V be the vector space of 2×2 real symmetric matrices.

(a) What is the dimension of V; prove it by exhibiting a basis for V.

Dim is 3. Basis $\underset{e_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$ $\underset{e_2}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$ $\underset{e_3}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$

(b) Let T be the linear transformation on V with $TM = M^T$ (thus a 2×2 matrix is sent to its transpose). Write the matrix of this T with respect to this basis (from (a)).

T is identity. Its matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

I accepted 2X2 identity as OK, since people may have been confused since we are taking about 2X2 matrices.

(25) 4. Consider the system of first-order differential equations

$$dv/dt = w$$
$$dw/dt = v.$$

(*a*) Write this in the form $u' = Au$ for an appropriate 2×2 matrix A (here u is a vector):

$u' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u$

char eqn:

$\lambda^2 = 1$

$\lambda = -1, \lambda = 1$

(*b*) Find eigenvalues and eigenvectors for A: indicate them *clearly* and show work.

$\lambda = -1$ evector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda = 1$ evector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(continued on next page)

(c) Find initial conditions $u(0) = u_0$ for some vector u_0 so that the solution decays at $t = \infty$. What is that solution?

So general solution is

$u = c \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c' \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$. Take $c = 0$.

So answer is

$$u = c \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

(20) 5. Let A be the matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix},$$

and show that the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite (you may use any test introduced in class).

• If we write $\mathbf{x}$ as $[x, y]^T$ with x and y scalars, write $\mathbf{x}^T A\mathbf{x}$ as a sum of squares (thus your expression will involve x and y).

Couple of ways

(1) principal minors: 1, $9-4=5$, both >0

(2) $x^2+4xy+9y^2 = (x+2y)^2+5y^2$

(some people had 2 here)

sum of squares

(3) Eigenvalues >0;

$\det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 9-\lambda \end{pmatrix} = \lambda^2-10\lambda+5$

roots $\lambda = +5 \pm \frac{1}{2}\sqrt{100-20}$

$= +5 \pm \sqrt{20}$

both roots >0

Note that in (2) we don't get the eigenvectors

(20) 6. Write down the Jordan form of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and explain in terms of properties of A (such as eigenvalues, nullspace...) how you arrived at your answer.

Find eigenvalues. They are easy since A is $\triangle$lar!
$\lambda = 1$ (mult 2), $\lambda = 0$. Or you can compute
$\det\begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}$ — but some students got the algebra wrong.

Then eigenvectors;

$\lambda = 0$; $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$; evector is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$: e_1

$\lambda = 1$ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$; evector is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$: e_2

So we know J has to be $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

The third basis vector is obtained by
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; vector is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

So M (not required) is $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Scratch

$x \leftarrow z$

0

z

$\begin{pmatrix} z \\ -y \\ 0 \end{pmatrix}$ (

$\begin{pmatrix} z \\ -y \\ 0 \end{pmatrix}$

Spread

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