3-1

Triangular factorization $A = LU$ or
$A = LDU$ (D: diagonal)

<u>if no row changes are needed</u>,
$Ax = b$
is transformed by left multiplication (elem. matrices) to
* $Ux = E_k E_{k-1} \cdots E_1 A x = E_k E_{k-1} \cdots E_1 b = c$,
(solve via back substitution).

$E_{ij}^{-1} = E_{ji}$. $E_{ij}(c)^{-1} = E_{ij}(-c)$, $E_i(c)^{-1} = E_i(c^{-1})$
To do this, we have to multiply by $E_1^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$
These are all lower Δular and, as we saw in the exercises, so $L = E_1^{-1} \cdots E_k^{-1}$. So we have for now

$A = LU$ (L^{-1} and L are lower Δlar, easy to multiply)

Book claims / show it — Note (p 34)
$A = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}$ cannot be factored as
$A = LU$. Why? $a_{11} = 0$ So
$l_{11} u_{11} + l_{12} u_{21} = 0$
But $l_{12} = 0$! So $l_{11} u_{11} = 0$. Suppose $l_{11} = 0$
Then the first row of L is zero — that means the first row of A is 0!

Let's review and count steps.
$Ax = b \iff LUx = b$,
and the right side gives a system!
* ¿where do we use no exchange of rows??

3-2

$$\begin{cases} Lc = b & \text{(gives } c\text{)} \text{ — start at top} \\ Ux = c & \text{— start at bottom} \end{cases}$$

(Solving Δlar systems is easier.

Note that there are two Equations

Note (p 14) when we count steps,

Operation: a division, and mult-subtraction ($E_{ij}(c)$) is one step. Let's count for $Ux = c$. Last row (to find x_n) is one operation, division by pivot element. Then $(n-1)$st row is 2 operations. Thus the total number is

$$1 + 2 + \dots + n = \frac{n(n+1)}{2} \quad \cancel{so}$$

and there are two.

Modification

$$A = LU = L'DU'$$

where L and U have 1's on the main diagonal. (L does already).

To get a matrix to which we may apply Gauss elimination without exchanging rows, we multiply on left by E_{ij} (this is the first time we use this). There are $n! = n(n-1)\dots 2 \cdot 1$ ways to rearrange rows, and each of them are a product of E_{ij}'s.

Note: saving L, U and (P = the E_{ij}'s) is useful.

Ex #7 p40. Look at 3rd row.

$$GH = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \qquad GF = \begin{bmatrix} 4 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$FGH = \begin{bmatrix} 0 & 2 & 1 & \boxed{0} \\ 0 & 0 & 2 & 1 \end{bmatrix} \qquad HGF = \begin{bmatrix} \boxed{4} & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{bmatrix}$$

4-1 (finish § 1.5; § 1.6)

P a permutation

The sign of P is $(-1)^k$, where k is the number of row exchanges, ~~Pf~~ In each row and in each column of P is exactly one "1", the other entries are zero.

1.6. Def of inverse, transpose: A^T $(a^T_{ij} = a_{ji})$

Exercise

$AA^{-1} = A^{-1}A = I$, $(AB)^T = B^T A^T$

Inverse exists if and only if elimination yields n pivots

Inverse is unique.

If we have A^{-1}, then can solve $Ax=b$.

If A is n×n, A^{-1} exists iff $Ax=0$ has only $x=0$ as solution

$(AB)^{-1} = B^{-1}A^{-1}$

$(ABC)^{-1} = ?$

Other definitions:

Symmetric,

If R is any matrix, R^TR and RR^T are symmetric

Do p 53 #17 to show $A = LDU$ is unique if A is invertible.

$L_1 D_1 U_1 = L_2 D_2 U_2$

$\underbrace{L_2^{-1} L_1 D_1}_{\text{lower}} = \underbrace{D_2 U_2 U_1^{-1}}_{\text{upper}}$

U ( ) D

Then compare diagonals

4-2

Gauss-Jordan for A^{-1}

Easy to see it we use matrix methods (you can multiply out by hand the first few times).

Find B so

$$AB = I$$

Write out for 3×3 or 4×4.

$$4\times4 \quad B = \underbrace{[e_1\ e_2\ e_3\ e_4]}_{I_4}$$

Display as

$$\left[\quad\right]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Mult both sides on left by elementary row operations.

$$\underbrace{Ax_1}_{\text{goes to } e_1} = e_1 \qquad (x_1 \text{ will be first column of } A^{-1})$$

→ Note that the elimination is the same for the Equation $Ax_1 = e_1$, $Ax_2 = e_2$, etc, since it depends only on A!

We do this (p 47) in stages

$$LA = U$$

then go from U to I by U^{-1} (mult. on left)

$$Ax = b$$

$$\boxed{Lc = b, \ Ux = c}$$

Note (p 49) we must have n pivots if A is n×n and nonsingular. Book asks, what happens if process breaks down? Suppose we get to stage

$$A = \begin{bmatrix}1&0&a&b\\0&1&c&d\\0&0&s&t\\0&0&0&u\end{bmatrix},$$

(start at bottom)

We claim we can't have a solution to

$$Ax = e_4$$

Try $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

If $u = 0$, this is clear, since $u \cdot x_4 = 1$ has no solution

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Then we have $u \neq 0$ and we show that if $s = 0$, the system

$$Ax = e_3$$

has no solution (since $s = 0$ there is not a full set of pivots).

$$\begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

We need $x_4 = 0$ for last row of Right Side, but then we can't get 3rd row to agree.