1. Geometric series. $\sum_{n=0}^{\infty} r^n \begin{cases} = \dfrac{1}{1-r}, & |r| < 1 \\ \text{diverges}, & |r| \geq 1. \end{cases}$

2. A necessary condition that $\sum_{n=1}^{\infty} a_n$ converge is that $\lim_{n\to\infty} a_n = 0$.

3. Integral Test. Suppose $f(x)$ is continuous, positive, and decreasing on $[a, \infty)$, for some $a \geq 1$. Then

$$\int_1^{\infty} f(x)\,dx \quad \text{and} \quad \sum_{n=1}^{\infty} f(n)$$

both converge or both diverge.

4. p**–series.**

$$\sum_{n=0}^{\infty} \frac{1}{n^p} \quad \begin{cases} \text{converges}, & p > 1 \\ \text{diverges}, & p \leq 1. \end{cases}$$

5. Comparison Test. Assume there is N_0 such that

$$0 < a_n \leq b_n, \quad n \geq N_0.$$

(i) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

6. Limit Comparison Test. Assume $a_n > 0$ and $b_n > 0$ for all n. Suppose

$$\lim_{n\to\infty} \frac{a_n}{b_n} = C, \quad \text{where} \quad 0 < C < \infty.$$

Then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

7. Alternating Series Test. Given $\sum_{n=1}^{\infty}(-1)^n a_n$, where $a_n > 0$. Suppose

(i) There is N_0 such that $a_n \geq a_{n+1}, \;\; n \geq N_0$.

(ii) $\lim_{n\to\infty} a_n = 0$.

Then the series converges. Moreover, if $S = \sum_{n=1}^{\infty}(-1)^n a_n$, then

$$|S - S_n| \leq a_{n+1}.$$

A series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges. It is **conditionally convergent** if it converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges.

8. Theorem. An absolutely convergent series is convergent.

9. Ratio Test. Given $\sum_{n=1}^{\infty} a_n$, with $a_n \neq 0$ for all n. If

(i) $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L < 1$, the series converges absolutely.

(ii) $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L > 1$, the series diverges.

10. Root Test. Given $\sum_{n=1}^{\infty} a_n$. If

(i) $\lim_{n\to\infty} |a_n|^{1/n} = L < 1$, the series converges absolutely.

(ii) $\lim_{n\to\infty} |a_n|^{1/n} = L > 1$, the series diverges.

11. Some useful limits.

(i) $\lim_{n\to\infty} \frac{n^p}{r^n} = 0, \quad \text{if } |r| > 1.$

(ii) $\lim_{n\to\infty} \frac{\ln n}{n^p} = 0, \quad \text{if } p > 0.$

(iii) $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$