4/21/2009 **MA 362** Answers

It is not necessary to complete all work, so long as when reading the paper I feel confident that all you have omitted is routine algebra, etc. Please write neatly so I can follow your work, that is part of your job. Be sure to justify asserts you make. I prefer pencil.

(20) 1(a). Finish the sentence: a (smooth) vector field $\mathbf{F}$ in $\mathbf{R}^3$ is *conservative* if:

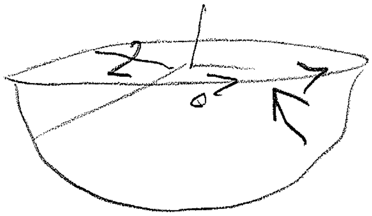
there is a (scalar) f with $\nabla f = F$

(b). Let $\mathbf{F}$ be as in (a). Derive the identity $\nabla \cdot (\nabla \times \mathbf{F}) = \mathbf{0}$.

Here $F = (P, Q, R)$ [not (f_x, f_y, f_z) !]

Computation to be made.

(20) 2. Let S be the lower hemisphere $x^2+y^2+z^2=4,\ z\le 0$, and $\mathbf{c}=\partial S$, oriented so that the (right-hand) normal to S points upward. Let $\mathbf{F}=(x+z, y+z, 2z)$. Stokes's theorem relates the integral of $\mathbf{F}$ on $\mathbf{c}$ (oriented as indicated) to a certain surface integral on S. Parametrize S and $\mathbf{c}$ and set up the two integrals which Stokes's theorem says are equal. **Do NOT evaluate these integrals!**



$$\nabla\times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x+z & y+z & 2z \end{vmatrix}$$

$$=(-1, 1, 0)$$

c: $(2\cos t, 2\sin t, 0)$

$$\int_c F\cdot ds = \iint_S \nabla\times F\cdot dS$$

I II

I: $\int_0^{2\pi} (2\cos t+0,\ 2\sin t+0, 0)\cdot(-2\sin t,\ 2\cos t, 0)\,dt$

II: $\nabla\times F = (-1,1,0)$

either S: $(x, y, -\sqrt{4-x^2-y^2})$ $x^2+y^2\le 4$

or S: $(2\cos u\sin v,\ 2\sin u\sin v,\ 2\cos v)$ $0\le u\le 2\pi$, $\pi/2\le v\le \pi$

Then write as

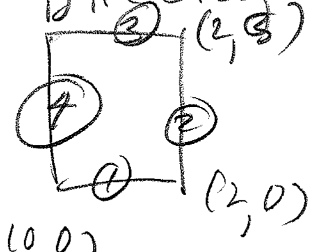
$$\iint (-1,1,0)\cdot(T_u\times T_v)\,du\,dv$$

(20) 3. Let D be the rectangle $0 \le x \le 2$, $0 \le y \le 3$, $\gamma = \partial D$. Consider $\int_\gamma y\,dx - 2x\,dy$. Explain how Green's theorem relates this to a certain area, and compute the line integral directly to see the Green's theorem gives the right answer.

Green says

$$\int y\,dx - 2x\,dy = -3\iint dx\,dy = -18$$

Directly:



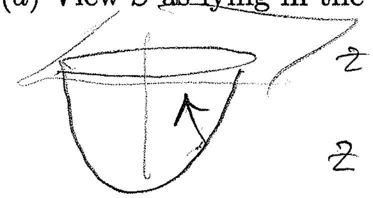
Four line integrals;

1. $\int_0^2 0\,dx - 2x\cdot 0 = 0$
2. $\int_0^3 y\cdot 0 - 4\,dy = -12$
3. $\int_2^0 3\,dx - 2x\cdot 0 = -6$
4. $\int_3^0 y\cdot 0 - 2\cdot 0\,dy = 0$

(20) 4. Let S be the parameterized surface given as

$$S = \Phi(u,v) = (u\cos v, u\sin v, u^2) \qquad 0 \le u \le 2,\ 0 \le v \le 2\pi.$$

(a) View S as lying in the xyz–space. Sketch S and its boundary.



$z = 4$

$z = r^2$

(b) Find the equation of the tangent plane to S at $(1,1,2)$. Is the normal you found pointing toward the z–axis or away from it?

$T_u = (\cos v, \sin v, 2u)$ $\qquad (1,1,2)$:

$T_v = (-\sin v\, u, \cos v \cdot u \cdot 0)$ $\qquad u = \sqrt{2},\ v = \frac{\pi}{4}$

$$T_u \times T_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 2u \\ -\sin v\, u & u\cos v & 0 \end{vmatrix}$$

$= (-2u^2\cos v, -2u^2\sin v, u)$ at $(\sqrt{2}, \frac{\pi}{4})$ $= \left(-\frac{4}{\sqrt{2}}, -\frac{4}{\sqrt{2}}, \sqrt{2}\right)$,

same direction as $(-4,-4,2)$ (component $z > 0$) points in

Plane $\quad -2(x-1) - 2(y-1) + (z-2) = 0$

(20) 5. Let $\mathbf{F} =< (1/2)z^2, 16\sin y, xz >$, and γ the curve (helix) $\gamma := (\cos t, \sin t, t^2)$, $0 \le t \le 1$. Find the work done by $\mathbf{F}$ on γ. (There is more than one way to do this!)

Substitute: $c: (\cos t, \sin t, t^2)$

$W = \int_c \left(\left(\frac{1}{2}\right)t^4, 16\sin(\sin t), t^2\cos t\right) \cdot (-\sin t, \cos t, 2t)\,dt$

(can be done directly).

<u>But</u>: $F = \nabla\left(\frac{1}{2}z^2 x - 16\cos y\right)$

curve joints $(1,0,0)$ to $(\cos 1, \sin 1, 1)$

so answer is

$\frac{1}{2}\cos 1 - 16\cos(\sin 1) - \left(\frac{1}{2}0^2 \cdot 1 - 16 \cdot 1\right)$