MA 362 ASSIGNMENT SHEET Spring 2009

Text: *Vector Calculus, J. Marsden and A. Tromba*, fifth edition

This sheet will be updated as the semester proceeds, and I expect to give several quizzes/exams. Calculus of several variables is both important in many applications of mathematics [finance, engineering, biology, ...] but also requires some sophisticated handling of abstract ideas. In principle, we are doing simple one-variable calculus, now with more than one independent variable, and our material has large intersection with our MA 261 [third-semester] calculus. However, past experience indicates that this is not an easy course. There will be homework collected most days, and it would help the discussion if you would e-mail me in advance asking for discussion of particular problems.

At the end of each lecture I will indicate the problems due for the next class, and it will be indicated on this sheet as it is updated.

It is important to come to **every** *class, and read the book at home. I will not be in class Tuesday, Jan. 20, and there will be an in-class major quiz during that period, based on what I cover the first week. I expect at least two other major exams, and depending on the flow of the class there may be additional quizzes.*

Some homework problems have answers/solutions in the back of the book. There are far too many problems for us to penetrate a good percent in class or have graded, but there are lots of opportunities for you to work out extra problems on your own. I will be glad to write out solutions and post them upon (reasonable) request, and after you give some effort on a problem without success, you can write to me (but be careful to describe the problem, as I might not have the textbook nearby).

In class I will do some extra problems, and they will also be considered as part of the basic course.

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Office Hours: T,θ 12:45-1:20, W 2:30-3 PM and by appointment. Feel free to email me too.

I am not likely to use the study guide very much.

There is an historical introduction which you should read on your own, and the authors incorporate a lot of interesting material using special fonts throughout the book. I am likely to ask a little about it, since mathematics is a very unusual subject – it is the same material all over the world, and has been an important part of our human heritage since ancient times.

It never hurts to try problems other than those assigned. Since this class is large, only a limited number of problems will be graded.

I do not intend to discuss Chapter 1, but please contact me if you want help on this (or any of the other material). (In fact, we are doing some of chapter 1, see below.)

2.3 *Differentiation.* The definition of the derivative we first encountered –

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

makes no sense when f is defined for a vector $\mathbf{x}$ instead of x. When the range of f is of more than one dimension, the derivative becomes a matrix, what the book calls the matrix of partial derivatives.

We will emphasize that having a differential means more than just having all partial derivatives, although examples to show this are rather non-intuitive.

We will go over the proofs of Theorems 8 and 9, and you should be able to understand them and reproduce at least parts of them. But even more important is to understand that a function may have partial derivatives at a point and not be continuous there!

We will see that the *mean-value theorem* from our one-variable calculus is far more important than we may have appreciated when we first saw it.

Problems: p. 139: 4bce, 5, 7d, 10, 12c, 16, 20.

1.3 (only a part) *Equation of a plane.* This is pages 50-53. A good test of understanding is if you can work out the formula for the distance from a plane to a point without memorizing (the Cosine is very important!).

Problems: p. 61: 9, 12, 16, 24, 29 (hint for 29: the equation of a plane P requires knowing a point and a vector perpendicular to the plane. We have many points on the plane given to us, and all we need is that the perpendicular to P be also perpendicular to $(3, 2, 4)$, the direction of the line. Of course there are infinitely many vectors perpendicular to $(3, 2, 4)$, but very few will work here. [1/20]

2.4 *Paths and curves.* Now the domain of the function is one dimension. This section should be routine, except that we will analyse the cycloid.

Problems: p. 149: 3, 4, 10, 13, 16, 20.

2.5 *Properties of the derivative.* You have seen these formulas in one-variable calculus, but in our setting we use linear algebra in many stages.

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Problems: p. 159: 2f, 3bc, 5bc, 6c, 10, 12, 16 (show in a picture where ∇f is at a point $(x, y) \neq (0, 0)$), 17b, 18, 1923, 24.

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2.6 **Gradients, directional derivatives, gradient field:**

Problems: p. 171: 2cd, 5c, 6ab, 12 (recall problem 16 of the previous set), 13c, 16, 20, 21.

3.1 *Mixed partial derivatives.* This is called Clairaut's theorem in our MA 261, at I don't think it is obvious.

Example: try $f(x, y) = |x|$ at the point $(0, 0)$.

Problems: p. 191: 11, 12, 16 (please show details), 19, 20abc, 21, 24.

3.2 *Taylor series.* We do up to second-order term, and describe remainder.
Problems: p. 202: 1, 2, 6.

3.3 *Local extrema.* We ues section 3.2 to classify critcal points (points where $\nabla f = 0$) in several variable, especially local maxima, minima and saddle points. The sign of the second term in the Taylor expansion tells the story.
Problems: p. 222: 5, 7, 8, 13, 19(!), 23, 24, 29.

3.4*Lagrange multipliers.* Principle: to extremize f subject to the single constraint $g = 0$,a point must satisfy $\nabla f \parallel \nabla g$. If instead f is to satisfy 2 constrains $g_1 = g_2 = 0$ (in three dimensions), then we must have a a point of local extremum that $\nabla f \cdot ((\nabla g_1) \times \nabla g_2)) = 0$.
Problems: p. 243: 2, 4, 7 (the situation with relative maximum requires a bit of thought, as we discussed in class), 12 (you can do this as in §3.3, but the Lagrange idea is far more elegant and clear!), 16; 20 is harded and is not required, but I'd be glad to discuss it with any student interested.

3.5 *Implicit Function theorem.* Here we have to be sensitive to the distinction between $f(\mathbf{x})$ and $f(\mathbf{x}) = 0$ (of course we can replace 0 by any constant c). In the latter case, there is the possibility for solving for one of the variables in terms of the others. You first saw that when you studied the equation $x^2 + y^2 = 1$, (likely in high school) and saw that near 'most' points on the graph, this may be replaced by either $y = \pm\sqrt{1 - x^2}$ or $x = \pm\sqrt{1 - y^2}$ (only one sign allowed). If the class knows Cramer's theorem, we may do the more general theorem on p. 251.
Problems: p. 253: 1, 3ab (you can do the first part using high-school algebra), 5.

4.1–4.3 *Vector-valued functions.* This should be pretty familiar from MA 261.
Problems: p. 273: 6, 7 [two ways means–you can either compute what is inside the [·] and then differentiate, or you can use the differentiation rules from page 262], 14– 16, 19a-c.
p. 282: 2, 4, 9, 11a-c (this shos that there is always a 'natural' parametrization of a path), 12, 13ab.

Vector fields, although no unfamiliar, will be important in the material which follows.
p. 293: 2, 4, 7, 8 [in these two, each vector has length one], 9, 11, 13, 14, 18 (differentiate!).

4.4 *Div and* ∇. There are a lot of identities on p. 306, and you have seen these notions in earlier courses. In this one, we hope to make clear their physical significance, but that does force us to wait a few chapters, until Chapter 8. The authors try their best in this section, however.

Problems: p. 311: 4, 5, 8, 12, 16, 18, 21, 24, 26 [there is a more direct way – if more complicated – to see this as well], 29.

Chapter 5. Double and triple integrals. We are not reviewing the thoery of double and triple integrals; I am assuming that you had these in your earlier courses, but if you have questions, please contact me. I am giving a selection of problems that you might consider, those with * should be handed in at least.

Problems: p. 347: 2cdf[all*][with sketches], 6, 8, 11*; p. 353: 2d*, 6*, 10*, 11; p. 365: 4*, 8*, 16, 19*, 30*.

6.1 *Change of variables.* We have in first-year calculus that if we integrate $\int_a^b f(x)\,dx$ and we change variables so that $x = g(t)$, that we get a t-integral by replacing dx by $g'(t)\,dt$. We want similar formulas in higher dimensions, but mappings are harder to see in two or more dimensions, since we can't make four- or 6-dimensional graphs!. So we introduce transformations here, and focus on the simples case that the transformation T is *linear*–what we study in linear algebra. (Here this means $Tx = Ax$, where A is a square matrix.

The main result in this section is that when A is linear and $\det(A) \neq 0$, then the image y of Ax [here x and y are vectors] with be a parallelogram if and only if the original x-object is a parallelogram.

Just as in one-variable calculus, these change-of-variables tricks can make integrals simpler to evaluate.

Problems: p. 375: 1, 4, 7, 8.

6.2 *The formula.* Most of this course depends on the idea that a 'smooth' map – this usually means that partial derivatives are continuous – behaves in small domains like a linear mapping. But we studied linear maps in §6.1, and what we see here is that (in a way that will be clear from the lecture):

$$dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv :$$

notice we have the *absolute value* of the Jacobian determinant here (this should have been anticipated by the discussion in class concerning the one-variable case for §6.1). We see integrals in polar and spherical coordinates too.

Problems: p. 390: 1, 2, 3 (this is a linear map), 6, 9, 18 [this is something like spherical coordinates, but not quite!], 24, 25, 32

6.3 *Applications.* Center of mass, average value, gravitational field.

Problems: p. 404: 2, 5, 12, 15, 16.

We now are coming to big-time! Line integrals, surface integrals are introduced so that we can understand the big theorems of Green, Gauss and Stokes (Gauss=divirgence!).

7.1, 2 *Path and line integrals.* We integrate wither scalar functions or vector fields on paths. Models to keep in mind (at least to me) are, respectively, the mass of a wire, and the work done by a force $\mathbf{F}$ as we go from two points in space (2, 3 or more dimensions).

Big points to keep in mind: (i) a line integral is "independent of parameterization" if we interpret it correctly (a curve should be covered only one, and unless we integrate with respect to ds, one might be off by a $-$ sign (this involves orientation matters).

Problems: p. 427: 1, 4, 6 [see discussion on p. 424]. 8ab, 9, 10, 12a.; p. 447: 2a-d, 4, 9, 11, 13 [not to be handed in, but we showed in class that the answer to the question is "no"], 16, 19ab [for a, the answer is that $df/dx = \|c'(x)\| = \sqrt{x'^2 + y'^2 + z'^2}$ at time $t = x$ (the x here is confusing, make sure you understand what I wrote here!).

7.3 *Parametrized surfaces.* When dealing with curves, we knew a long time ago that not all curves in the plane are readily exhibited as $y = f(x)$; we need in general $x = f(t),\ y = g(t)$ and (maybe) $z = h(t)$. The same issue comes with surfaces. Think of the sphere: we don't always have $z = f(x, y)$. In general we use parametrized surfaces: $\Phi : (, v) \to (x(u, v), y(u, v), z(u, v))$, where (x, y) range over some region D in the plane. Do this for a sphere or cone, why is $z = f(x, y)$ also an example of this? Φ is supposed to be at least differentiable, or maybe C^1; the book is not too clear on this.

The vector $T_u \times T_v$ is perpendicular to S – at least when it is not the zero vector! The book also uses $\mathbf{n}$ to be $T_u \times T_v$, as a 'normal' vector, but of course in this formulation it need not have length one, and it ignores when $T_u \times T_v = 0$.

The book uses the notion of *regular* surface as one for which $t_u \times T_v$ is never 0.

Problems: p. 459: 2 [the point $(-1/4, 1/2, 2)$ refers to the x, y, z–coordinates; you have to interpret that information in terms of u and v], 5, 8, 12 [solve a quadratic, we are told $z > 0$], 14, 16, 17.

7.4 *Surface area.* The texpression $T_u \times T_v$ is defined only up to $\pm$ sign. So we define the surface area of a parametrized surface S by

$$A(S) = \int\int_D \|T_u \times T_v\|\, dudv,$$

where $\Phi : D \to S$. This definition is really tricky to work out in real generality, and the book gives some history (but on p. 469 'this century' means 'the last century').

Problems: p. 471: 1 (work out the formula for the area of a sphere–that is the point of this problem), 2 (what portions of the sphere(s) do we get?), 5 (what is this surface), 7 (this involves improper integrals, we integrate over $y^2 + z^2 < R$ (this is a proper integral), and let $R \to \infty$), 9 (think of the equation of a sphere as a model), 10, 14, 17.

7.5 *Integrals of scalar funtions on surfaces.* Just plug in. We want

$$\int\int_S f(x,y,z)\,dS.$$

This becomes an integral with respect to $dudv$, much as we saw with line integrals. In the specail case that S is given as $z = f(x,y)$, we have that $dS = \sec\gamma dxdy$, where γ is the angle between the tangent plane to S at a point and the vertical axis. Why is it natural that this factor $\sec\gamma$ always be > 1?

Problems. p. 481: 3, 5, 6 (important formula), 9 (can you choose coordinates cleverly?), 10, 11, 12a, 15ab (to be used later in the chapter), 18 (this is not easy, see me if you wish!).

7.6 *Integrating vector fields on surfaces.* we plug in, but what is new is the we need to choose an *orientation* for S, much as when we look at a (physical) line, our paramatrization will give a choice of positive direction. Not all surfaces have an orientation, as Möbius tells us. If $z = f(x,y)$ we have the upward-pointing or downward-pointing normals. Notice that if $T_u \times T_v \neq 0$, we get a 'natural' orientation.

Vocabulary: flux

Problems. p. 497: 2, 4, 7, 8a (don't compute the integral if too complicated), 11 (make a picture), 13, 14, 17.

7.7 *Some applications.* Curvature of a surface: mean curvature and gauss curvature (this is genuinely new material to almost all of you!). we mention the Gauss-Bonnet theorem. It asserts that although the (Gauss) curvature depends on a parametrization,

$$\int\int_S K dA$$

depends just on the 'topology' of S–if we deform S that integral will not change!

Problems. p. 513: 1 (we learn the term 'minimal sureface' here–some new ones have only been discovered in your lifetimes!), 4, 7, 9.

8.1 *Green's theorem.* This is given for the plane, but in 3 dimensions appears as the divirgence theorem (Gauss's theorem) and Stoeks's theorem.

Problems p. 528: 2, 4 ['directly' means that you are to compute the line integral], 7, 8 [include a picture!], 10, 19. Please look at number 12–compute both sides of Green's theorem and see that the two sides do not agree. We will discuss this in class.

8.2 *Stokes's theorem.* This is the theorem

$$\int\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F}\ cdot\, d\mathbf{s}.$$

It is presented in several guises (nonparametric surfaces, parametric surfaces) and then we have an interpretation of *circulation*.

Problems. p. 547: 2, 3 [so you compute both sides!], 5 [you should think about what surface(s) you want to use], 11, 15 (this should impress you!), 17, 19, 21.

8.3 *Conservative fields, when is* $\mathbf{f} = \nabla f$*?* Don't forget to compare with Ex. 12, p. 529 in either two or three dimensions, which looks like a counterexample!

We also have Theorem 8, p. 558: if $\mathbf{F}$ os a C^1 vector field defined on all of space and div $\mathbf{F} = 0$, then there is a C^1 vector field $\mathbf{G}$ with $\mathbf{F} = \nabla \times \mathbf{G}$.

Problems. p. 558: 1, 2, 7, 9, 12 (again!), 14 (all)17, 18.

8.4 *Gauss' theorem; divergence theorem.* I think this is easier to explain and understand (such as if S is the boundary of a box) than Stokes's theorem:

$$\int\int\int_W (\nabla \cdot \mathbf{F})\, dV = \int\int_{\partial W} \mathbf{F} \cdot d\mathbf{S}.$$

This gives a real interpretation of divirgence, far more physical than the formula we reviewed some chapters ago, and which you learned in second-year calculus.

The book derives expressions for divirgence in spherical coordinates using Gauss's theorem. We can do that for other computations as well. This is good technique, and you should not try to memorize the formulas, only the principles.

Problems. p. 573: 2, 5, 7, 11 (hint: first rearrange the equation), 14, 15, 18 ('prove' here is meant that you should be able to explain why the equation is true), 21, 23.

(4/7/09)