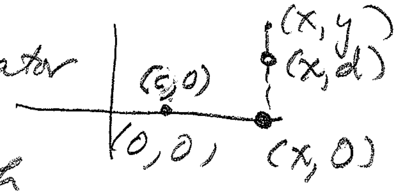
Proof of Theorem 2.9 for special case that $f:\mathbb{R}^2\to\mathbb{R}$,
We assume the partials of f (with respect to both x and y) are continuous near the point $\underline{x}_0=(0,0)$.

We need to show (by definition of differentiability) that

$$(*)\qquad \frac{f(x,y)-f(0,0)-f_x(0,0)(x-0)-f_y(0,0)(y-0)}{\sqrt{(x-0)^2+(y-0)^2}}\to 0$$

as $(x,y)\to(0,0)$.



Let's just analyze the numerator of $(*)$ for now, going from (x,y) to $(0,0)$ by first letting $y\to 0$ with <u>x fixed</u>, and then $x\to 0$ with y fixed at $y=0$ (see figure). On these vertical and horizontal segments we can apply the <u>mean-value theorem</u>, and so we have

$$(A)\quad f(x,y)-f(0,0)=\big(f(x,y)-f(x,0)\big)+\big(f(x,0)-f(0,0)\big)$$
$$=y\frac{\partial f}{\partial y}(x,d)+x\frac{\partial f}{\partial x}(c,0),$$

where c and d come from the <u>mean-value theorem</u>.

<u>Now</u> we use that f_x and f_y are <u>continuous</u> near $(0,0)$. Since $(x,y)\to(0,0)$ here, the points (x,d) and $(c,0)$ are near $(0,0)$. Thus

$$(B)\quad f_y(x,d)=f_y(0,0)+\eta_2,\qquad f_x(c,0)=f_x(0,0)+\eta_1,$$

where $\eta_1\to 0$ as $c\to 0$ and $\eta_2\to 0$ as $(x,d)\to(0,0)$ [of course, as x changes, d changes, but this does not matter here].

This means that the numerator of $(*)$ is (using (A) and (B))

$$f(x,y)-f(0,0)-f_x(0,0)\,x-f_y(0,0)=$$
$$y\big(f_y(0,0)+\eta_2\big)+x\big(f_x(0,0)+\eta_1\big)-x\,f_x(0,0)-y\,f_y(0,0)$$
$$=\eta_1 x+\eta_2 y.$$

So the full expression in $(*)$ reduces to

$$\eta_1\frac{x}{\sqrt{x^2+y^2}}+\eta_2\frac{y}{\sqrt{x^2+y^2}}.$$

Since $|x/\sqrt{x^2+y^2}|\le 1$ and $|y/\sqrt{x^2+y^2}|\le 1$ and $\eta_1,\eta_2\to 0$ we are done!