2/19/2009 **MA 510** Solutions

I am anxious to see how well you understand the material, and so it is not necessary to complete all work, so long as when reading your paper I feel confident that all you have omitted is routine algebra, etc. Please write neatly so I can follow your work, that is part of your job. You should always give reasons when you make assertiions. I prefer pencil.

(20) 1. Let $w = f(x, y)$, where $x = u + 2v)$ and $y = u - 2v$. Using the chain rule, show that

$$w_{uv} = 2[f_{xx} - f_{yy}].$$

Thus you should first compute w_u and then differentiate that expression with respect to v.

$$w_u = f_x x_u + f_y y_u = f_x + f_y$$

$$w_{uv} = [f_{xx} x_v + f_{xy} y_v] + [f_{yx} x_v + f_{yy} y_v]$$

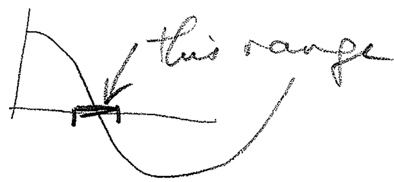
$$= f_{xx} 2 - 2 f_{xy} + 2 f_{yx} - 2 f_{yy}$$

(20) 2(a). A vehicle is on a hill given by the equation $z = 2x^2 + y^2$ at $(1, 2)$, and runs at the fixed speed of 10. Unfortunately, the vehicle will be destroyed if it goes up a slope of more that .1. Describe the possible directions from which the vehicle can proceed from $(1, 2)$. I want to see a computation with an answer as well as a drawing showing the point $(1, 2)$ together with the possible set of directions from there. (Hint: first find the direction at which the height increases most rapidly (or decreases most rapidly) your final answer should probably involve that direction.) $\nabla z = (4, 4)$ at $(1,2)$

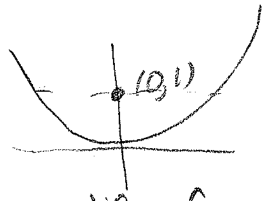
$\frac{dz}{dt}$ (in direction u) $= D_u z \cdot v$, where v is the velocity. Since $D_u z = \nabla z \cdot u = |\nabla z| \cdot 1 \cdot \cos\theta$, we have the situation

$$4\sqrt{2} \cdot \cos\theta \cdot 10 \le 1/10$$

$$\cos\theta \le \frac{1}{4\sqrt{2} \cdot 100} = \frac{\sqrt{2}}{800}$$



(20) 3. Find the closest point on the ~~plane~~ parabola $x^2 = 4y$ to the point $(0, 1)$. To do this, you must to identify a function to be extremized, and then use calculus to find the possible point(s). Check that you have a minimum, not a maximum.



Minimize the function

$$(x-0)^2 + (y-1)^2 = x^2 + y^2 - 2y + 1$$

But $x^2 = 4y$, so we have to minimize

$$4y + y^2 - 2y + 1 \;;\; y^2 + 2y + 1 = f(y)$$

$f'(y) = 2y + 2$, so $f'(y) = 0$ when $y = -1$.

But $y \geq 0$. Since this is a parabola opening up, the minimum is when $y = 0$ (we have $0 \leq y \leq \infty$, and $f \uparrow \infty$ at $y \to \infty$)

(20) 4. For what values A does $f(x, y) = 3x^2 + 2y^2 + Axy$ have a saddle point at the origin?

$$H = \begin{pmatrix} 6 & A \\ A & 4 \end{pmatrix}$$

so saddle if $24 - A^2 < 0$,

$A > \sqrt{24}$ or $A < -\sqrt{24}$

(20) 5. Consider the system of equations

$$F: \quad xu + yv^2 = 12$$
$$G: \quad 2xu^2 + y^2v = 80$$

near the point $(x, y, u, v) = (2, 4, 4, 1)$. We are interested in whether near this point the system can be expressed in either of the simpler forms $u = u(x, y)$, $v = v(x, y)$ or $x = x(u, v)$, $y = y(u, v)$. The implicit function theorem gives an answer, please explain which (or maybe both) is/are possible and why.

To solve for x, y need $\frac{\partial(F,G)}{\partial(x,y)} \neq 0$
— — u, v — $\frac{\partial(F,G)}{\partial(u,v)} \neq 0$ } at $(2,4,4,1)$.

But $\frac{\partial(F,G)}{\partial(x,y)} = \begin{pmatrix} u & v^2 \\ 2u^2 & 2yv \end{pmatrix}$ and at $(2,4,4,1)$ this is $\begin{pmatrix} 4 & 1 \\ 32 & 8 \end{pmatrix}$ — determinant zero, can't solve.

On the other hand
$\frac{\partial(F,G)}{\partial(u,v)} = \begin{pmatrix} x & 2yv \\ 4xu & y^2 \end{pmatrix}$ at $(2,4,4,1)$ is $\begin{pmatrix} 2 & 8 \\ 32 & 1 \end{pmatrix}$ det $\neq 0$

so OK, can solve.