3/10/2009 **MA 510** Solutions (sketchy)

It is not necessary to complete all work, so long as when reading the paper I feel confident that all you have omitted is routine algebra, etc. Please write neatly so I can follow your work, that is part of your job. You should always give reasons when you make assertiions. I prefer pencil.

(15) 1. (Chain rule.) Let $g(t) = f(\mathbf{x} + t\mathbf{h})$, where $\mathbf{x} = (x_1, x_2)$, $\mathbf{h} = (h_1, h_2)$. Find $g'(t)$ and $g''(t)$.

$$g'(t) = \nabla f \cdot \frac{d}{dt}(\underline{x} + t\underline{h}) = (f_{x_1}, f_{x_2}) \cdot \underline{h} = h_1 f_{x_1} + h_2 f_{x_2}$$

For $g''(t)$ we get

$$(*) \quad g''(t) = h_1 (f_{x_1})' + h_2 (f_{x_2})'$$

bad notation

and, by the first part,

$$(f_{x_1})' = h_1 (f_{x_1})_{x_1} + h_2 (f_{x_1})_{x_2},$$

$$(f_{x_2})' = h_1 (f_{x_2})_{x_1} + h_2 (f_{x_2})_{x_2}.$$

$\boxed{g'(t) = h_1 f_{x_1} + h_2 f_{x_2}}$

Combining with $(*)$,

$$g''(t) = h_1 (h_1 f_{x_1 x_1} + h_2 f_{x_1 x_2}) + h_2 (h_1 f_{x_2 x_1} + h_2 f_{x_2 x_2})$$

$$= h_1^2 f_{x_1 x_1} + 2 h_1 h_2 f_{x_1 x_2} + h_2^2 f_{x_2 x_2}$$

(15) 2. (Flow lines). Let $\mathbf{F}$ be the vector field $\mathbf{F} = (y, -x, z)$, and let $\mathbf{c}$ be a flow line so that $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$. Suppose that at time $t = 0$, $\mathbf{c}(0) = (1,1,1)$ and in general $\mathbf{c}(t) = (x(t), \cos t, z(t))$ (so that the y-component is known). Find $x(t)$ for all time t. (We don't worry about $z(t)$.)

The problem states that we have the system;

$c'(t) = F(c(t))$, $c(0) = (1,1,1)$

$$\begin{cases} x'(t) = y(t), & x(0) = 1 \\ y'(t) = -x(t), & y(0) = 1 \\ z'(t) = z(t), & z(0) = 1 \end{cases}$$

But we are told $y(t) = \cos t$. So to find $x(t)$, we have

$x'(t) = \cos t$, $x(t) = -\sin t + c$, and since $x(0) = 1$, $c = 1$,

$\therefore \boxed{x(t) = -\sin t + 1}$

(20) 3. Let the vector field on $R^3 \setminus \{0\}$ be given by $\mathbf{F} = \mathbf{r}/r^3$ ($r = \sqrt{x^2+y^2+z^2}$). Show that $\mathbf{F}$ is conservative by finding a potential V for $\mathbf{F}$.

We want $-\nabla V = \dfrac{(x, y, z)}{(x^2+y^2+z^2)^{3/2}}$

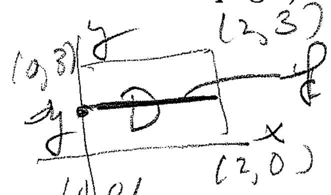
$V = -\dfrac{1}{\sqrt{x^2+y^2+z^2}}$ $\left(\text{so } \dfrac{\partial V}{\partial x} = -\dfrac{x}{(\sqrt{x^2+y^2+z^2})^3}, \text{ etc.}\right)$

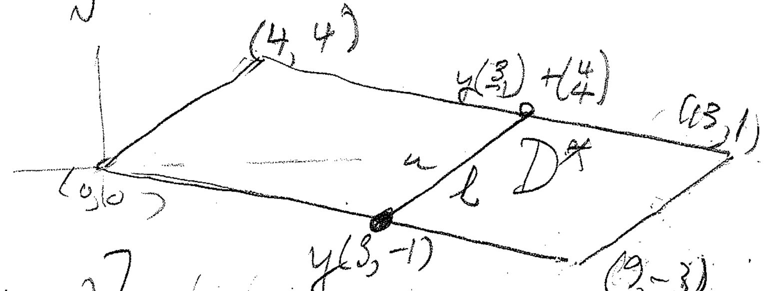
(20) 4(*a*). Consider the transformation

$$u = 2x + 3y$$
$$v = 2x - y.$$

Let D be the rectangle in the x, y–plane with vertices at $(0,0)$, $(2,0)$, $(0,3)$ and $(2,3)$, and D^* its image in the u, v–plane.

We proved in class that D^* is a parallelogram. Verify this by letting ℓ be the segment connecting $(0, y)$ to $(2, y)$, where $0 \le y \le 3$, and show that its image ℓ^* in the u, v–plane is a line (this means you must be able to give the parametric equation of a line). (This is continued on the next page.)





$0 \le t \le 1$

$\ell: (0,y) + t[(2,y) - (0,y)]$

$x = 2t$
$y = y$

So $u = 4t + 3y$ (y is fixed)
$v = 4t - y$

or $\begin{pmatrix} u \\ v \end{pmatrix} = y\begin{pmatrix} 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} 4 \\ 4 \end{pmatrix}$, $0 \le t \le 1$, $0 \le y \le 3$ a line.

Note: $\dfrac{\partial(u,v)}{\partial(x,y)} = -8$

(b).

By inspection
$4y = u - v$
$4x + 2y = u + v$

Write down

$$\int\int_D e^{2y}\sin(4x+2y)\,dxdy$$

as an integral in the u, v–plane. Show limits. You need not compute the integral.

$$\int_0^3\int_0^2 e^{\frac{1}{2}(u-v)}\sin(u+v)\,(1/8)\,du\,dv$$

abs value of Jacobian
$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$

(15) 5. Let Ω be the cylinder $x^2 + y^2 \leq 4,\ 1 \leq z \leq 4$, with density $\delta(x, y, z) = 5(x^2 + y^2)z^2$. The z–coordinate of the center of mass of Ω, $\bar{z}$ is the ratio of two integrals. Write down this ratio using *cylindrical coordinates.*}

$$\bar{z} = \frac{\int_1^4 \int_0^{2\pi} \int_0^2 z \cdot 5r^2 z^2 \, r \, dr \, d\theta \, dz}{\int_1^4 \int_0^{2\pi} \int_0^2 5r^2 z^2 \, r \, dr \, d\theta \, dz}$$