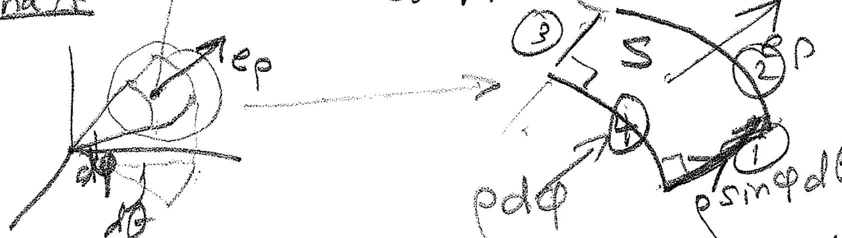
Curl in other systems.

Let $F = F_\rho \underline{e}_\rho + F_\theta \underline{e}_\theta + F_\varphi \underline{e}_\varphi$ be a vector field. Find $\nabla \times F$ in terms of $e_\rho, e_\theta, e_\varphi$, as on p 543. Write

$$\Delta \times F = A e_\rho + B e_\theta + C e_\varphi,$$
$$\uparrow \nabla \times F$$

$$\text{So } A = (\nabla \times F)\cdot e_\rho$$

Find A



By Stokes (waving hands only a little)!

$$\iint_S \Delta \times F \cdot d\underline{S} = \int_{\partial F} F \cdot ds = \int_{①} + \int_{②} + \int_{③} + \int_{④} F \cdot ds.$$

Now the LHS $= \iint (\Delta \times F)\cdot e_\rho\, dS \simeq A \iint dS = A\rho^2 \sin\varphi\, d\theta\, d\varphi$

For the RHS!

① ! Since e_θ is in the direction of the ~~tangent~~ on circle,

$$\int_{①} F \cdot d\underline{s} \sim \int (F \cdot e_\theta) \cdot \frac{ds}{\text{length}} = F_\theta(\rho, \theta, \varphi)\, \rho \sin\varphi\, d\theta,$$

$$\int_{②} F\, ds \underset{\sim}{(\text{direction} - e_\varphi)} \int F \cdot e_\varphi\, ds = -F_\varphi(\rho, \theta, \varphi)\, \rho\, d\varphi.$$

On ③, φ has changed by $-d\varphi$ from ①, and on ④, θ has <u>decreased</u> by $d\theta$. The direction of ③ is $-e_\theta$, and that of ④ is e_ρ. Thus

$$\int_{③} F \cdot d\underline{s} \sim -F_\theta(\rho, \theta, \varphi - d\varphi)\, \rho \sin(\varphi - d\varphi)\, d\theta$$

$$\int_{④} F \cdot ds \sim F_\varphi(\rho, \theta - d\theta, \varphi)\, \rho\, d\varphi.$$

Thus RHS

$$= \big(F_\theta(\rho, \theta, \varphi)\sin\varphi - F_\theta(\rho, \theta, \varphi - d\varphi)\sin(\varphi - d\varphi)\big)\rho\, d\theta$$
$$- \big(F_\varphi(\rho, \theta - d\theta, \varphi) - F_\varphi(\rho, \theta, \varphi)\big)\rho\, d\varphi$$

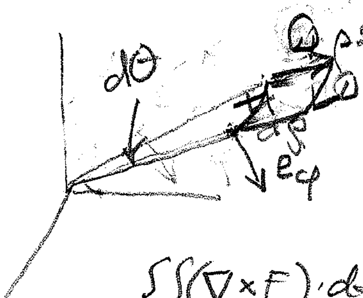
Thus using the mean value theorem,

$$A(\rho^2 \sin\varphi\, d\theta\, d\varphi) = \left(\frac{\partial}{\partial\varphi}(F_\theta(\rho, \theta, \varphi)\sin\varphi) - \frac{\partial}{\partial\theta} F_\varphi(\rho, \theta, \varphi)\right)\rho\, d\theta\, d\varphi!$$

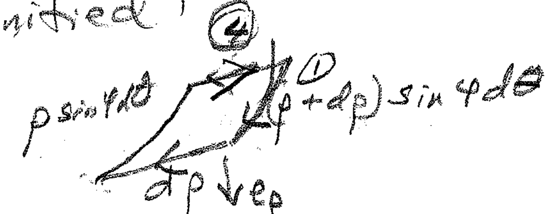
$$A = \frac{1}{\rho \sin\varphi}\left(-\frac{\partial}{\partial\theta} F_\varphi + \frac{\partial}{\partial\varphi}(F_\theta \sin\varphi)\right)$$

(Check on ① + ③ $g(x) - g(x-h) \approx h\, g'(x)$!

Let's do the coefficient C, for e_φ.



Magnified:



$$\iint_S (\nabla\times F)\cdot dS = \iint (\nabla\times F)\cdot e_\varphi\, dS = \int_{\partial S} F\cdot d\underline{s} \qquad (e_\varphi \text{ points } \underline{\text{down}})$$

LHS $\approx C\, d\rho\, \rho\sin\varphi\, d\theta$

RHS? We add ① - ④ again. (I do ① then ③ then ② then ④)

On ① $d\underline{s} = -e_\theta(\rho+d\rho)\sin\varphi\, d\theta$, so

$$\int_{①} F\cdot ds \sim -F_\theta(\rho+d\rho,\theta,\varphi)\,(\rho+d\rho)\sin\varphi\, d\theta,$$

$$\int_{③} F\cdot ds \approx F_\theta(\rho,\theta,\varphi)\,\rho\sin\varphi\, d\theta,$$

$$\int_{②} F\cdot ds \approx -F_\rho(\rho,\theta,\varphi)\, d\rho,$$

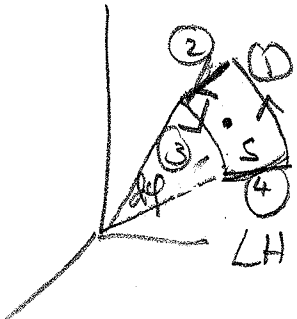
$$\int_{④} F\cdot ds = F_\rho(\rho,\theta+d\theta,\varphi)\, d\rho$$

$$\therefore \left(\int_{①}+\int_{③}\right)+\left(\int_{②}+\int_{④}\right) = -\frac{\partial}{\partial\rho}\left(F_\theta(\rho,\theta,\varphi)\cdot\rho\right)\sin\varphi\, d\theta\, d\rho$$

$$+\ \frac{\partial}{\partial\theta}F_\rho(\rho,\theta,\varphi)\, d\rho\, d\theta$$

$$\therefore\quad C = \frac{1}{\rho\sin\varphi}\left[\frac{\partial F_\rho}{\partial\theta} - \sin\rho\,\frac{\partial}{\partial\rho}(\rho F_\theta)\right]$$

Set-up for B:



(e_θ points into the ~~sheet~~)

φ changes $\quad$ ρ changes

$$\iint(\nabla\times F)\cdot e_\theta\, dS = \int_{①}+\int_{③}+\int_{②}+\int_{④}$$

LHS $\approx B(\rho\, d\varphi)\rho\sin\varphi\, d\theta$

$$\int_{①} F\cdot ds \approx -F_\varphi(\rho+d\rho,\theta,\varphi)(\rho+d\rho)\, d\varphi \qquad (\varphi \text{ is decreasing on ①})$$

$$\int_{②} F\cdot ds = -F_\rho(\rho,\theta,\varphi-d\varphi)\, d\rho \qquad (\rho \text{ is decreasing})$$

(rest left as exercise!)