4/21/2009 **MA 510** ANSWERS/remarks

It is not necessary to complete all work, so long as when reading the paper I feel confident that all you have omitted is routine algebra, etc. Please write neatly so I can follow your work, that is part of your job. You should always give reasons when you make assertions. I prefer pencil.

(20) 1(*a*). A vector field $\mathbf{F}$ on $\mathbf{R}^3$ is *conservative* if: there is a (scalar) function f with $\nabla f = \mathbf{F}$

(*b*). Let f be a smooth function in a domain $D \subset \mathbf{R}^3$. Prove that $\nabla \times (\nabla f) = \mathbf{0}$.

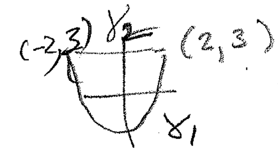
$\nabla f = (f_x, f_y, f_z)$ (so there is a function f and these are its partial derivatives)

$$\nabla \times \nabla f = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = (f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy})$$

(*c*). Is the converse true? (An example, indication of proof or good discussion is adequate.)

No. We saw in class that $\frac{r}{\|r\|}$ has curl 0, but is not the gradient of any function.

(Converse means: if $\nabla \times G = 0$, is G the gradient of some scalar function f — relates to paths)



(20) 2. Let D be the region bounded by $y = x^2 - 1$ and $y = 3$, and γ the boundary of D oriented in the positive direction. Consider $\int_\gamma x^2 + y\,dx$. Compute this line integral using Green's theorem and directly.

a) Green's theorem

$$\int_\gamma x^2+y\,dx \overset{(\text{Green})}{=} \iint_{\text{inside } \gamma} -1\,dx\,dy = -\int_{-2}^{2}\int_{x^2-1}^{3} dy\,dx = -\int_{-2}^{2} 4-x^2\,dx$$

$$= -4x\Big|_{-2}^{2} + \frac{1}{3}x^3\Big|_{-2}^{2} = -\left(16 - \frac{1}{3}(8-(-8))\right)$$

$$= -\frac{32}{3}$$

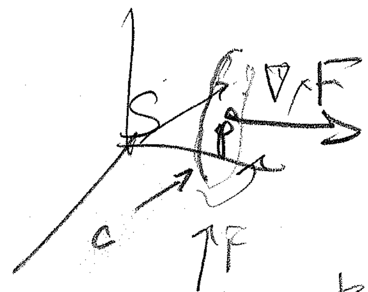
b) Directly: $\gamma_1 : (x, x^2-1)$ $(-2 \le x \le 2)$, $\gamma_2 : (x, 3)$ $(-2 \le x \le 2)$, so

$$\int_\gamma x^2+y\,dx = \int_{-2}^{2} (x^2+(x^2-1))\,dx + \int_{2}^{-2} x^2+3\,dx$$ ← note direction

$\uparrow y$

$$= \int_{-2}^{2} x^2-4\,dx$$ (then compare with the ←)

(20) 3. Let **F** be a smooth vector field, and at $\mathbf{p} = (1, 4, 2)$ suppose that $\nabla \times \mathbf{F}(\mathbf{p}) = 3\mathbf{j}$. Use Stokes's theorem to describe **F** as concretely as possible near **p**. Make an illustration to help explain your answer.



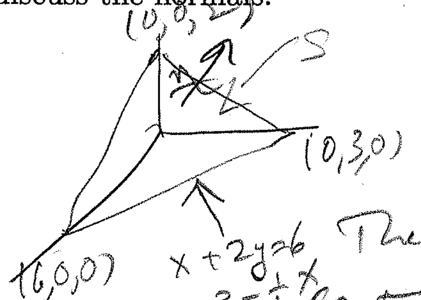
(by Stokes, we take a surface S whose normal is in the direction of ∇×F at p. Then

$$\iint \nabla \times F \cdot d\underline{S} = \int_{\text{bdy } S} F \cdot ds$$

If $C = \partial S$, and C is oriented so that the normal to S with determined by following C with the right hand, then since $\iint \nabla \times F \cdot dS > 0$, there is net circulation about C from F.

Do not confuse F and $\nabla \times F$ here!

(20) 4. Let S be the portion of the plane $x + 2y + 3z = 6$ with $z \geq 0$. Let $\mathbf{F}$ be the vector field $\mathbf{F} =< x - z, 2z, z - y >$. Sketch S and set up (no need to evaluate) $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ as an ordinary double integral with limits (which you need not evaluate). Be sure to discuss the normals.



$\iint F \cdot dS$

$z = 2 - \frac{1}{2}y - \frac{1}{3}x$

$d\underline{S} = \pm\left(-\frac{1}{3}, \frac{1}{2}, 1\right)$

(use the normal pointing up)

Then $y = 3 - \frac{1}{2}x$

$\iint F \cdot dS$

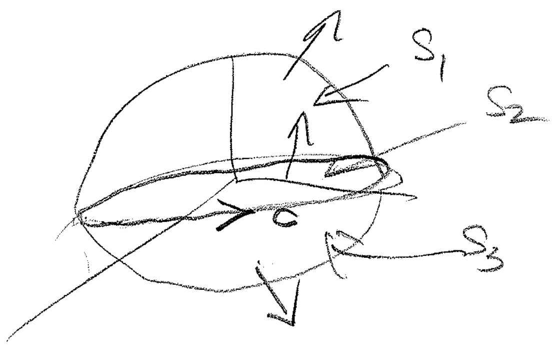
$$= \iint \left(x - \left(2 - \tfrac{1}{2}y - \tfrac{1}{3}x\right), 2\left(2 - \tfrac{1}{2}y - \tfrac{1}{3}x\right), 2 - \tfrac{3}{2}y - \tfrac{1}{3}x\right) \cdot \left(\tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{2}\right) dx\,dy$$

$$= \int_0^6 \int_0^{3-\frac{1}{2}x} \tfrac{1}{3}\left(x - \left(2 - \tfrac{1}{2}y - \tfrac{1}{3}x\right)\right) + \left(2 - \tfrac{1}{2}y - \tfrac{1}{3}x\right) + \tfrac{1}{2}\left(2 - \tfrac{3}{2}y - \tfrac{1}{3}x\right) \, dy\,dx$$

(20) 5. Let $\mathbf{F}$ be a smooth vector field defined inside the sphere of radius 5 centered at the origin. Let S_1 be the upper hemisphere $x^2 + y^2 + z^2 = 4, z \geq 0$, S_2 the disk $x^2 + y^2 \leq 4, z = 0$; and S_3 be the lower hemisphere $x^2 + y^2 + z^2 = 4,\ z \leq 0$. With S any of these, discuss the possible values of

$$\int\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S},$$

not ignoring issues of orientation. Simple sketches may help explain.



I want answers in terms of

$\int_C F \cdot ds$,

with C oriented as shown.

The simple answer is that these surface integrals will all be the same if we choose the normals to each of them to all point up or point down. Otherwise they might differ by a factor of (-1).