MA 510 ASSIGNMENT SHEET Spring 2009

Text: Vector Calculus, J. Marsden and A. Tromba , fifth edition

This sheet will be updated as the semester proceeds, and I expect to give several quizzes/exams. the calculus of several variables is both important in many applications of mathematics nance, engineering, biology , ...] but also requires sophisticated handling of abstract ideas. It is used unbiquitously by engineers and other scientists. There will be homework collected most days, and it would help the discussion if you would e-mail me in advance asking for discussion of particular problems.

It is important to come to **every** *class, and read the book at home. I will not be in class Tuesday, Jan. 20, and there will be an in-class major quiz during that period, based on what I cover the first week.*

Some of the homework problems have answers/solutions in the back. There are far too many problems for us to penetrate a good percent in class, but there are lots of opportunities for you to work out problems on your own. I will be glad to write out solutions and post them upon (reasonable) request.

In class I will do some extra problems, and they will also be considered as part of the basic course.

Since the backgrounds of the students here is so varied, we will start at the beginning of the text, but move quickly.

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Office Hours: T,θ 12:45-1:20, W 2:30-3 PM and by appointment. Feel free to email me too.

I am not likely to use the study guide very much.

There is an historical introduction which you should read on your own. I am likely to ask a little about it, since mathematics is a very unusual subject – it is the same material all over the world, and has been an important part of our human heritage since ancient times.

It never hurts to try problems other than those assigned. Since this class is large, only a limited number of problems will be graded.

There is a lot of material, and I would like to have time for the final chapters(s), which often get rushed. So we may not cover all sections in this outline.

At the end of each class, I will indicate which problems are due for the next class, and this will then be updated on this outline.

1.1–2, 1.5(part) *Introduction and later highlights.* Most of the course concerns things you may have seen (at a less careful level) before, but Chapter 1 is especially elementary.

We will work in n dimensions from the beginning. One thought: when considering a vector, $\mathbf{x}$, it doesn't matter if we think of $\mathbf{x} = (x_1, x_2, x_3)$ or $(x_1, x_2, \ldots, x_n)$.

I teach vector subtraction a little differently than you see elsewhere.

Note: I insist that you get used to vector notation as soon as possible: at first a vector will be written $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, but soon it will be $\mathbf{x}$, and then just x.

Problems: p. 21: 5, 11, 14, 18, 19, 23, 25, 28. [* due 1/15]

p. 36: 12, 14, 18, 22, 26 (make this clean by using dot product). [due 1/15]
p. 86: 2 (all), 5, 9, 15. [due 1/15]

1.3–1.5 (rest) *Matrices etc.* A matrix is an $n \times n$ array of 'scalars', but when $m = n$ we can do more arithmetic. It seems to me that what makes several variables more difficult than one variable is that we need to use matrices rather than numbers/scalars, and as a result a lot of 'natural' manipulations we'd like to perform make no sense. This especially will appear in section 2.3.

While the dot product of two n-dimensional vectors always makes sense, the cross product works only in three dimensions.

Thought: you should not have to memorize the formula on page 53 for the distance from a point to a plane.

Problems: p. 61: 1, 5, 15c, 17, 21c, 23, 27 (try to give a clean solution), 31, 36, 38 (this insight is important for linear algebra, try to explain what is happening).
p. 73: 2, 3, 6b, 8, 15 (hint: do an algebraic operation with the equation first).
p. 86: 11, 12, 13, 17, 18.

2.1 *Real-valued functions.* In the past we studied $y = f(x)$, where y and x were real numbers. Now we let one of x, y, or both, be vectors. Of course, this means that graphing will be even less important than in earlier courses, but we still can get insights from it. We will be doing max-min, but a new ingredient is the possibility of a saddle-point, see. Ex. 6, p. 103.

Problems: p. 105: 2 (all), 6, 7, 10, 14, 17, 18 [you should use some words in answering the last three], 21-26.

2.2 **Limits and Continuity.** Don't be fooled, this is almost word-for-word what you had in one-vairable calculus. However, we need the notion of open, closed sets, and their boundaries(why is the situation far easier in one variable?). We will build some technique for evaluating limits; this is a math course! I expect we will do a little $\varepsilon - \delta$ work. Examples 14 and 15 show how it is far harder to determine if limits exist, and in fact we will rarely see these types of functions later.

Problems: p. 125: 3, 4, 5, 6 [in class I discuss when you can 'plug in' to find a limit], 8ab, 11; 14, 15b, 17, 18, 19, 21 [try to write one down, using perhaps $\|\|$].

2.3 *Differentiation.* This is the first place where we develop new intuition. We will see that the one-variable difinition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The easiest concept to understand is the the partial derivative, but the notion of being *differentiable* is really the main point. Having a differential is the correct analgue of having a derivative with functions of more than one independent variable (the range can be one or many variables). Having partial derivatives at a point does not even guarantee that the function is continuous. We are likely to carefully do the proof of Theorem 9 (that of Theorem 8 is much more elementary).

Problems: p. 139: 4bce, 5, 7d, 10, 12c, 16, 20.

2.4 *Paths and curves:* Now the domain of the function is one-dimension. This section should be routine, except that we will analyse the cycloid.

Problems: p. 149: 3, 4, 10, 13, 16, 20.

2.5 *Properties of the derivative:* You have seen these in one-variable calculus, but here we need to use linear algebra in many stages.

Problems: p. 159: 2f, 3bc, 5bc, 6c, 10, 12, 16 (show in a picture where ∇f is at a point $(x, y) \neq (0, 0)$), 17b, 18, 1923, 24.

2.6 *Gradients, directional derivatives, gradient field.*

Problems: p. 171: 2cd, 5c, 6ab, 12 (recall problem 16 of the previous set), 13c, 16, 20, 21.

3.1 *Clairaut and Taylor.* I don't think it obvious that $f_{xy} = f_{yx}$ if f has two continuous derivatives. A counterexample is $f(x, y) = |x|$ (why does that not contradict my first sentence?). We also introduce the fundamental PDEs of mathematical physics, at least some of which will be derived at the end of the course.

Problems: p. 191: 6, 11, 12, 16, 19 (any thoughts on the origin of the word 'harmonic' here?), 20, 21, 24 (another 'counterexample' to Caliraut).

3.2 *Taylor theorem.* This is used by engineers and its derivation here is a good use of the chair rule.

Problems. p. 202: 4-6.

3.5 *Implicit Function Theorem.* The problem faced here was first encountered in high school. We learned the definition of a 'function' $y = f(x)$, and then discovered that the cirle : $x^2 + y^2 = 1$ is not a function! All we can do is resolve it either as one of $y = \pm\sqrt{1 - x^2}$ or $x = \pm\sqrt{1 - y^2}$, and this section shows what really is happening in general. The general implicit function theorem and inverse function theorems are very sophisticated forms of this.

Problems: p. 253: 1, 3 [you can sove this without calculus in part a], 5, 6, 8, 9, 10-12.

4.1 *Vector-valued functions.* When the domain is one-dimensional, the math is simpler, no matter how many dimensions is the range. We do this chapter very quickly, since it should be review.

Problems: p. 273: 5, 6 [by 'two ways', this means either we first do the operation with what is inside the [ ], and then compare it (after differentiating) with the rules on p. 262], 12, 14 [clever calculus proof wanted!], 15, 19a-c.

4.2 *Arc length, parameterization.*
p. 281: 2, 7 (can you sketch this curve, at least for $0 \leq t \leq 2\pi$?), 10–13.

(*Vector fields*) p. 293: 3, 6, 7, 8 (note for these last two each vector has length one),11, 16, 18 (differentiate), 19.

4.4 *Div and* ∇. The physical significance of these is the subject chapter 8, but the authors try a bit here. Theorems 1 and 2 just use Clairaut's theorem; a full list of these is on p. 306.

Problems: p. 310: 6, 8, 16, 28, 30, 31.

Chapter 5 *Double and Triple Integrals.* This chapter is largely for self-study; see me if you did not learn this material as an undergraduate.

Problems: p. 365: 1, 5, 10, 14, 25, 27, 28, 29, 33, 34, 36.

6.1 *Maps* $\mathbb{R}^2 \to \mathbb{R}^2$. we need to understand changing of variables, since an important tool is to phrase formulas using coordinates in which they are most natural.

Problems: p. 375: 1, 2, 4, 7, 8, 9 (here the transformation is simply given by matrix multiplication).

6.2 *The change-of-variables theorem and jacobians.* We see how we integrate in polar, cylindrical and spherical coordinates.

Problems: p. 390: 1, 3ab, 8, 12, 15, 21, 25, 29.

6.3 *Applications.* We sketch the center of mass and gravitational fields.

Problems: p. 404: 3, 5, 8, 12, 15, 16.

7.1-2 *Line/Path integrals.* We now take integrals with respect to dx, dy, dz or ds. We either integrate scalar functions (such as if we want to find the mass of a wire) or vector functions (finding work or *flux*). In one case we integrate a scalar-valued function, in the other case a vector-valued function.

In the problems for this section, we are just substituting into an integral. Your 'childish' notion that an integral is some kind of 'area' will not help you here– an integral is just a Riemann sum.

Be sure you understand that the Theorem on page 440 is simply the Fundamental Theorem of Calculus. dr and $d\mathbf{r}$.

Problems: p. 427: 1 [to do this, please look at the Riemann sums as on the bottom of p. 423, and take the limit as the number of divisions tends to ∞], 2a, 4ab, 8 [for c, just do it for what is in 2a], 13, 15.

p. 447: 2bd, 4, 6, 9, 13 [we discussed this is class a few weeks ago], 16, 18 [the questions about what is unrealistic is a good one, think about it!], 191-c.

7.3 *Parametrized surfaces.* (recall our discussions of parametrized curves.) If you remember speherical coordinates, you can find a good parametrization of a sphere of radius R; what happens if we just work with the equation $x^2 + y^2 + z^2 = R^2$? Instead of a tangent line to a curve, we get a *tangent plane* to our surface.

The surface will be *regular* if $T_u \times T_v$ is never 0.

Problems: p. 459: 2, 5, 6 [what is the surface?], 8, 12, 15 [by now, you should be good at vector calculus!], 17 [d is the hard one, try it–we also see why it helps to have our surface be regular...].

7.4 *Surface area.* To do this carefully is not easy, but for nice functions we have useful formulas. There are two formulas for surface area of a surface S, depending on whether S is a parametrized surface or a graph (but they can be obtained in one swoop).

Problems: p. 471: 1, 2 [problem 1 came up an an earlier homework problem], 6, 7 [this requires a bit about improper integrals, and we skipped that section, so see me if you are not sure about this one!], 9 (Hint: for parametric representation, think of a sphere), 10, 15, 16, 20, 21.

7.5 *Integrals of scalar functions over a surface* S. We compute $\int\int_S f(x, y, z)\, dS$ by 'changing veriables.' Notice that $\|T_x \times T_v\|$ comes into this. When we do this for the situation $z = f(x, y)$ and notice a relation between $\|T_u \times T_v\|\, dudv$ and $dxdy$–understand the factor $\sec\gamma$, where *gamma* is the angle between the normal to S and the z-axis.

Problems. p. 480: 2, 6 (useful formula), 8, 12abc, 14, 15ab.

7.6 *Surface integrals.* We integrate a vector function and so consider

$$\int\int_S \mathbf{F} \cdot d\mathbf{S}.$$

When we change variables here, the order in which we choose u and v will determine a *sign* of this integral. That is one good reason for wanting regular surfaces. *Flux* is introduced on p. 491. Understand the physical interpretation. Summary is on p. 496.

Problems. p. 497: 2, 4, 5, 9, 10, 18 (good preparation for the next chapter).

7.7 *Appllication.* We learn about mean curvature and Gauss curvature, measuring hos $T - u$ and T_v change with the normal. It is a very abstract concept, but some of the most important mathematicians have thought about this. There is a lot of history in the text for this section. I will try to explain the significance of the Gauss-Bonnet theorem. It explains why some integrals really depend only in a very small way on the surface S–if we deform S is any (nonviolent!) way, the integral won't change.

Problems p. 513: 1 (minimal surfaces; some new ones were introduced in your livetimes!), 4, 7, 9.

8.1 *Green's theorem.* This is the two-dimensional form of the fundamental theorem of calculus, and it is surprisingly closely related. We also see that it is a pregnant form of both the divirgence theorem and Stokes's theorem. Note the role of the exterior normal in the divirgence theorem.

Problems. p. 528: 1, 4, 5, 6b, 7, 11, 13.

8.1 *more!* We discuss problem 12 and exercises 21-29 from this section. Some will be assigned as homework.

8.2 *Stokes's theorem.* It is stated for nonparametized surfaces and parametrized surfaces. We also get an interpretation of the curl, and Faraday's law.

Problems. p. 547: 1-3 (by 'verify' we mean computing both sides of Stokes's theorem), 5 (read and think about this carefully!), 13, 15 (important observation), 17, 19, 25.

8.3 *Conservative fields, when with* $\mathbf{f} = \nabla f$*?* Don't forget to compare with Ex. 12, p. 529 in either two or three dimensions, which looks like a counterexample!

We also have Theorem 8, p. 558: if $\mathbf{F}$ os a C^1 vector field defined on all of space and div $\mathbf{F} = 0$, then there is a C^1 vector field $\mathbf{G}$ with $\mathbf{F} = \nabla \times \mathbf{G}$.

Problems. p. 558: 1, 2, 7, 9, 12 (again!), 14 (all)17, 18.

8.4 *Gauss' theorem; divirgence theorem.* I think this is easier to explain and understand (such as if S is the boundary of a box) than Stokes's theorem:

$$\int\int\int_W (\nabla \cdot \mathbf{F}) = \int\int_{\partial W} \mathbf{F} \cdot d\mathbf{S}.$$

This gives a real interpretation of divirgence, far more physical than the formula we reviewed some chapters ago, and which you learned in second-year calculus.

The book derives divirgence in spherical coordinates using Gauss's theorem. We can do that for other computations as well. This is good technique, and you should not try to memorize the formulas, only the principles.

Problems. p. 573: 2, 5, 7, 11 (hint: first rearrange the equation), 14, 15, 18 ('prove' here is meant that you should be able to explain why the equation is true), 21, 23.

8.5 *Applications.* Conservation laws, Euler's equation for a perfect fluid, Heat equation, Maxwell's equations. An avalanche of applications.

Problems. p. 586: 1-3, 7a, 10ab.

(4/7/09)