Contents

1. Exam 1

1. (10 pts) Determine a lower bound for the radius of convergence of series solutions

$$(x^2 - 4x + 5)y'' + (x + 3)y' + 4(x^2 - 4x + 5)y = 0$$

about $x_0 = 1$.

2. Consider the differential equation

$$(x - 2)^2(x + 1)y'' + 3(x^2 + x - 6)y' + (4x + 1)y = 0.$$

(a) (8 pts) Show that $x_0 = 2$ is a regular singular point.
(b) (7 pts) Find the indicial equation of a series solution of the form

$$y = \phi(r, x) = \sum_{n=0}^{\infty} a_n(r)(x - 2)^{r+n}$$

and also find the exponents at the singular point $x_0 = 2$.

3. Consider a series solution $y = \sum_{n=0}^{\infty} a_n x^n$ about $x_0 = 0$ of

$$y'' - xy' - 2y = 0.$$

(a) (10 pts) Find the recurrence relation for a_n.
(b) (5 pts) Find a general formula for a_n.
(c) (5 pts) Find two linearly independent series solutions.

4. (15 pts) Use the Laplace transform to solve the initial value problem

$$y'' - 7y' + 12y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

5.

(a) (8 pts) Use the definition of the Laplace transform

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt$$

to show that

$$\mathcal{L}\{e^{at}\} = \frac{1}{s - a} \qquad \text{for } s > a.$$

(b) (7 pts) Find the Laplace transform of

$$f(t) = \left[2(t - 5)^2 + \cos(t - 5) + 4\right] u_5(t).$$

6. (10 pts) Find the inverse Laplace transform of

$$F(s) = \frac{(s + 1)e^{-7s}}{s^2 - 6s + 13}.$$

7. Consider the initial value problem

$$(\dagger) \quad \phi'(t) - \int_0^t (t - \xi)^2 \phi(\xi) d\xi = \delta(t - 3), \quad \phi(0) = 1.$$

(a) (8 pts) Convert the differential equation (†) to an algebraic equation in $\Phi(s) = \mathcal{L}\{\phi(t)\}$ (but do **not** solve the equation).
(b) (7 pts) Let $\phi(t)$ be the solution of the equation (†). Evaluate the following integral

$$\int_0^{\infty} e^{-st}(2\phi(t) + \sin 3t)dt.$$

2. Exam 1-Solution

1. (10 pts) Determine a lower bound for the radius of convergence of series solutions

$$(x^2 - 4x + 5)y'' + (x+3)y' + 4(x^2 - 4x + 5)y = 0$$

about $x_0 = 1$.

Solution) Note that $x_0 = 1$ is an ordinary point and

$$p(x) = \frac{x+3}{x^2 - 4x + 5}, \quad q(x) = 4.$$

The roots of $x^2 - 4x + 5 = 0$ are $2 + i$ and $2 - i$. The distance from $x_0 = 1$ to the nearest root $2 + i$ is $\sqrt{2}$ (you may take $2 - i$ as well), and so the radius of convergence of $p(x)$ is $\rho_p = \sqrt{2}$. The radius of convergence of $q(x)$ is $\rho_q = \infty$. Therefore, $\min\{\sqrt{2}, \infty\} = \sqrt{2}$ and so we find that the radius of convergence ρ of the series solution is at least $\sqrt{2}$, which is a lower bound.

2. Consider the differential equation

$$(x-2)^2(x+1)y'' + 3(x^2 + x - 6)y' + (4x+1)y = 0.$$

(a) (8 pts) Show that $x_0 = 2$ is a regular singular point.

Solution) Clearly, $x_0 = 2$ is a singular point because

$$(x-2)^2(x+1) = 0 \implies x = 2, -1.$$

Use $x^2 + x - 6 = (x-2)(x+3)$ and compute

$$p_0 = \lim_{x\to 2}(x-2)p(x) = \lim_{x\to 2}(x-2)\frac{3(x-2)(x+3)}{(x-2)^2(x+1)} = \lim_{x\to 2}\frac{3(x+3)}{(x+1)} = 5$$

and

$$q_0 = \lim_{x\to 2}(x-2)^2 q(x) = \lim_{x\to 2}(x-2)^2\frac{4x+1}{(x-2)^2(x+1)} = \lim_{x\to 2}\frac{4x+1}{x+1} = 3.$$

Therefore $x_0 = 1$ is a regular singular point.

(b) (7 pts) Find the indicial equation of a series solution of the form

$$y = \phi(r, x) = \sum_{n=0}^{\infty} a_n(r)(x-2)^{r+n}$$

and also find the exponents at the singular point $x_0 = 2$.

Solution) The indicial equation is

$$0 = r(r-1) + p_0 r + q_0 = r(r-1) + 5r + 3 \implies r^2 + 4r + 3 = 0 \implies r = -1, -3$$

and the exponents of singularity at $x_0 = 2$ are $r = -1, -3$.

3. Consider a series solution $y = \sum_{n=0}^{\infty} a_n x^n$ about $x_0 = 0$ of

$$y'' - xy' - 2y = 0.$$

(a) (10 pts) Find the recurrence relation for a_n.
(b) (5 pts) Find a general formula for a_n.
(c) (5 pts) Find two linearly independent series solutions.

Solution) (a) Compute

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

and put y, y', y'' into the differential equation to find that

$$\begin{aligned} 0 &= y'' - xy' - 2y \\ &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} na_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} na_n x^n - 2\sum_{n=0}^{\infty} a_n x^n \end{aligned}$$

Use the shifting formula $n \to n+2$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

to get

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} na_n x^n - 2\sum_{n=0}^{\infty} a_n x^n \\ &= (2a_2 - 2a_0) + \sum_{n=1}^{\infty}\Big[(n+2)(n+1)a_{n+2} - (n+2)a_n\Big] \end{aligned}$$

and

$$\begin{cases} a_2 = a_0 \\ (n+2)(n+1)a_{n+2} - (n+2)a_n = 0, \quad n \geq 1 \end{cases}$$

(b) The recurrence relation can be simplified to

$$a_{n+2} = \frac{1}{n+1}a_n, \quad n \geq 1.$$

Considering *even and odd* cases we see that

$$a_{2m} = \frac{a_0}{1 \cdot 3 \cdot 5 \cdots (2m-3) \cdot (2m-1)}, \quad m \geq 1$$

and

$$a_{2m+1} = \frac{a_1}{2 \cdot 4 \cdot 6 \cdots (2m-2) \cdot (2m)}, \quad m \geq 0.$$

(c) The general solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + \sum_{m=1}^{\infty} a_{2m}x^{2m} + \sum_{m=0}^{\infty} a_{2m+1}x^{2m+1} \\ &= a_0 + \sum_{m=1}^{\infty} \frac{a_0}{1 \cdot 3 \cdot 5 \cdots (2m-3) \cdot (2m-1)}x^{2m} + \sum_{m=0}^{\infty} \frac{a_1}{2 \cdot 4 \cdot 6 \cdots (2m-2) \cdot (2m)}x^{2m+1} \end{aligned}$$

and so

$$y_1(x) = 1 + \sum_{m=1}^{\infty} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2m-3) \cdot (2m-1)}x^{2m}$$

and

$$y_2(x) = \sum_{m=0}^{\infty} \frac{1}{2 \cdot 4 \cdot 6 \cdots (2m-2) \cdot (2m)}x^{2m+1}$$

are two linearly independent solutions.

4. (15 pts) Use the Laplace transform to solve the initial value problem

$$y'' - 7y' + 12y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution) Set $Y(s) = \mathcal{L}\{y(t)\}$ and compute the Laplace transform

$$\begin{aligned} 0 &= \mathcal{L}\{y'' - 7y' + 12y\} \\ &= \left[s^2Y(s) - sy(0) - y'(0)\right] - 7\left[sY(s) - y(0)\right] + 12Y(s) \\ &= \left[s^2Y(s) - s\right] - 7\left[sY(s) - 1\right] + 12Y(s) \\ &= (s^2 - 7s + 12)Y(s) - s + 7. \end{aligned}$$

Solve for $Y(s)$ and compute the partial fractions

$$Y(s) = \frac{s+3}{s^2 - 7s + 12} = \frac{s-7}{(s-3)(s-4)} = \frac{4}{s-3} - \frac{3}{s-4}.$$

Taking the inverse Laplace transform we find that

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} = 4e^{3t} - 3e^{4t}.$$

5.

(a) (8 pts) Use the definition of the Laplace transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

to show that

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad \text{for } s > a.$$

Solution) By the definition

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at}dt = \int_0^\infty e^{-(s-a)t}dt = \lim_{A\to\infty}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^A = \lim_{A\to\infty}\frac{e^{-(s-a)A}}{-(s-a)} + \frac{1}{s-a} = \frac{1}{s-a}$$

because $\lim_{A\to\infty} e^{-kA} = 0$ for $k = s - a > 0$.

(b) (7 pts) Find the Laplace transform of

$$f(t) = \left[2(t-5)^2 + \cos(t-5) + 4\right]u_5(t).$$

Solution) Note that

$$f(t) = h(t-5)u_5(t)$$

where

$$h(t) = 2t^2 + \cos t + 4 \implies H(s) = \mathcal{L}\{h(t)\} = \frac{4}{s^3} + \frac{s}{s^2+1} + \frac{4}{s}.$$

By the general formula

$$\mathcal{L}\{f(t)\} = e^{-5s}H(s) = e^{-5s}\left(\frac{4}{s^3} + \frac{s}{s^2+1} + \frac{4}{s}\right).$$

6. (10 pts) Find the inverse Laplace transform of

$$F(s) = \frac{(s+1)e^{-7s}}{s^2 - 6s + 13}.$$

Solution) Write

$$F(s) = \frac{(s+1)e^{-7s}}{s^2 - 6s + 13} = e^{-7s}H(s)$$

where

$$H(s) = \frac{(s+1)}{s^2 - 6s + 13} = \frac{s-3}{(s-3)^2 + 4} + 2\frac{2}{(s-3)^2+4} \implies h(t) = \mathcal{L}^{-1}\{H(s)\} = e^{3t}\cos 2t + 2e^{3t}\sin 2t$$

from $s^2 - 6s + 13 = (s-3)^2 + 4$. We see that

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{e^{-7s}H(s)\} = h(t-7)u_7(t) = \left[e^{3(t-7)}\cos 2(t-7) + 2e^{3(t-7)}\sin 2(t-7)\right]u_7(t).$$

7. Consider the initial value problem

$$(\dagger) \quad \phi'(t) - \int_0^t (t-\xi)^2\phi(\xi)d\xi = \delta(t-3), \quad \phi(0) = 1.$$

(a) (8 pts) Convert the differential equation (†) to an algebraic equation in $\Phi(s) = \mathcal{L}\{\phi(t)\}$ (but do **not** solve the equation).

Solution) Let $\Phi(s) = \mathcal{L}\{\phi(t)\}$, $f(t) = t^2$ and $F(s) = \mathcal{L}\{f(t)\} = \frac{2}{s^3}$. Note that we may rewrite the equation (†) as follows:

$$\phi'(t) - (f * \phi)(t) = \delta(t-3).$$

Take the Laplace transforms on both sides to see

$$\mathcal{L}\{\phi'(t)\}+\mathcal{L}\{(f*\phi)(t)\} = \mathcal{L}\{\delta(t-3)\} \implies (s\Phi(s)-\phi(0))-F(s)\Phi(s) = e^{-3s} \implies \left(s-\frac{2}{s^3}\right)\Phi(s) = 1+e^{-3s}.$$

(b) (7 pts) Let $\phi(t)$ be the solution of the equation (†). Evaluate the following integral

$$\int_0^\infty e^{-st}(2\phi(t)+\sin 3t)dt.$$

Solution) Note that

$$\int_0^\infty e^{-st}(2\phi(t)+\sin 3t)dt = 2\mathcal{L}\{\phi(t)\}+\mathcal{L}\{\sin 3t\} = 2\Phi(s)+\frac{3}{s^2+9} = \frac{2s^3(1+e^{-3s})}{s^4-1}+\frac{3}{s^2+9}$$

because

$$\left(s-\frac{2}{s^3}\right)\Phi(s) = 1+e^{-3s} \implies \Phi(s) = \frac{s^3(1+e^{-3s})}{s^4-2}$$

from (a).