Contents

1. Exam 2

1. (10 pts) Consider the initial value problem

$$\begin{cases} y' = 2t + 4y, \\ y(2) = 4. \end{cases}$$

Use the Euler method

$$y_{n+1} = y_n + hf(t_n, y_n)$$

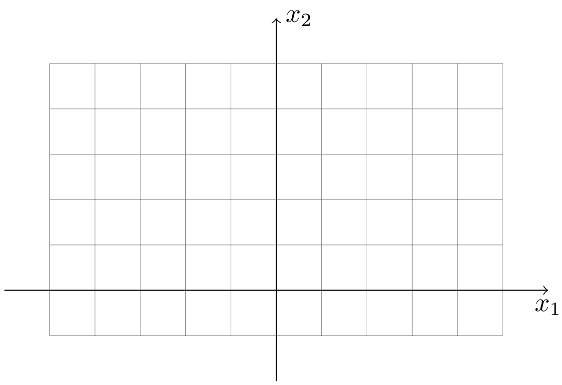
with step size $h = 0.1$ to approximate $y(2.1)$.

2.

(a) (7 pts) Draw the tangent vector (an arrow) of the solution (curve) of the system

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{x}$$

at $\mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ in the following coordinates:



(b) (8 pts) The general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{x}$$

is given by

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Find the solution of the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

3. (15 pts) The general solution of the system

$$(*) \quad \mathbf{x}' = \begin{pmatrix} 2t^{-1} & -t^{-1} \\ 3t^{-1} & -2t^{-1} \end{pmatrix} \mathbf{x}, \quad t > 0$$

is given by

$$\mathbf{x}(t) = c_1 \begin{pmatrix} t \\ t \end{pmatrix} + c_2 \begin{pmatrix} t^{-1} \\ 3t^{-1} \end{pmatrix}.$$

Assume that a particular solution $\mathbf{x}(t)$ of the nonhomogeneous system

$$\mathbf{x}' = \begin{pmatrix} 2t^{-1} & -t^{-1} \\ 3t^{-1} & -2t^{-1} \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2t^2 \\ 4t \end{pmatrix}$$

is of the form $\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{u}(t)$ where $\mathbf{\Psi}(t)$ is a fundamental matrix of the associated homogeneous system $(*)$. Use the method of variation of parameters to find $\mathbf{u}(t)$.

4. (15 pts) Consider the linear system

$$\mathbf{x}' = \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}}_{A} \mathbf{x}.$$

Then

$$\det(A - \lambda I) = -(\lambda - 3)^2(\lambda - 1) = 0,$$

and linearly independent eigenvectors are given as follows:

$$\left[\lambda = 3 \implies \boldsymbol{\xi} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right], \quad \left[\lambda = 1 \implies \boldsymbol{\xi} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]$$

Find the general solution of the system.

5. Consider the initial value problem

$$\begin{cases} y' = \frac{2009y^2+4t^2}{3+3t^2}, \\ y(5) = 3. \end{cases}$$

The Runge-Kutta method is given as follows:

$$y_{n+1} = y_n + (h/6)(k_{n_1} + 2k_{n_2} + 2k_{n_3} + k_{n_4})$$

where

$$\begin{aligned} k_{n_1} &= f(t_n, y_n) \\ k_{n_2} &= f(t_n + (1/2)h, y_n + (1/2)hk_{n_1}) \\ k_{n_3} &= f(t_n + (1/2)h, y_n + (1/2)hk_{n_2}) \\ k_{n_4} &= f(t_n + h, y_n + hk_{n_3}). \end{aligned}$$

(a) (5 pts) What is the meaning of the global truncation error E_n?
(b) (5 pts) The general theory says that the Runge-Kutta method is a fourth order approximation method. What does "fourth order" mean?
(c) (5 pts) Choose the step size $h = 0.2$. Find n for which y_n approximates $y(5.8)$ when we use the Runge-Kutta method.

2. Exam 2-Solution

1. (10 pts) Consider the initial value problem

$$\begin{cases} y' = 2t + 4y, \\ y(2) = 4. \end{cases}$$

Use the Euler method

$$y_{n+1} = y_n + hf(t_n, y_n)$$

with step size $h = 0.1$ to approximate $y(2.1)$.

Solution) First, note that

$$y(2) = 4 \implies t_0 = 2, \quad y_0 = 4.$$

Since $t_1 = t_0 + h = 2.1$ we need to compute y_1. Set $n = 0$ in the general formula $y_{n+1} = y_n + hf(t_n, y_n)$:

$$f(2,4) = 2 \cdot 2 + 4 \cdot 4 = 20 \implies y_1 = y_0 + hf(t_0, y_0) = 4 + (0.1)f(2,4) = 2 + (0.1)20 = 6.$$

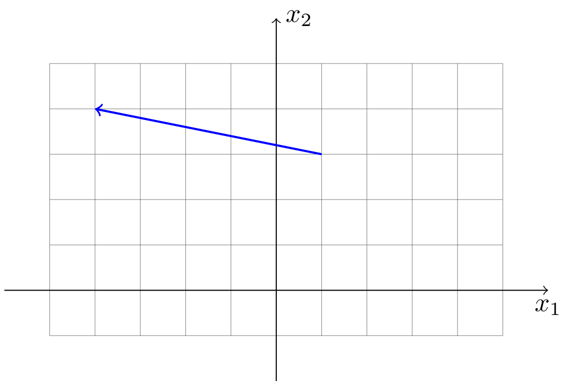
We find that $y_1 = 6 \approx y(2.1)$.

2.

(a) (7 pts) Draw the tangent vector (an arrow) of the solution (curve) of the system

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{x}$$

at $\mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ in the following coordinates:



Solution) The tangent vector at at $\mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}.$$

We need to sketch the vector $(x_1', x_2') = (-5, 1)$ at $(x_1, x_2) = (1, 3)$ and the answer is shown above.

(b) (8 pts) The general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{x}$$

is given by

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Find the solution of the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Solution) The initial condition says

$$t = 0: \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \implies \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \implies \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and so the solution is

$$\mathbf{x}(t) = 2e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2e^{-t} + e^{3t} \\ 2e^{-t} + 3e^{3t} \end{pmatrix}.$$

3. (15 pts) The general solution of the system

$$(*) \quad \mathbf{x}' = \begin{pmatrix} 2t^{-1} & -t^{-1} \\ 3t^{-1} & -2t^{-1} \end{pmatrix} \mathbf{x}, \quad t > 0$$

is given by

$$\mathbf{x}(t) = c_1 \begin{pmatrix} t \\ t \end{pmatrix} + c_2 \begin{pmatrix} t^{-1} \\ 3t^{-1} \end{pmatrix}.$$

Assume that a particular solution $\mathbf{x}(t)$ of the nonhomogeneous system

$$\mathbf{x}' = \begin{pmatrix} 2t^{-1} & -t^{-1} \\ 3t^{-1} & -2t^{-1} \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2t^2 \\ 4t \end{pmatrix}$$

is of the form $\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{u}(t)$ where $\mathbf{\Psi}(t)$ is a fundamental matrix of the associated homogeneous system $(*)$. Use the method of variation of parameters to find $\mathbf{u}(t)$.

Solution) A fundamental matrix is

$$\mathbf{\Psi}(t) = \begin{pmatrix} t & t^{-1} \\ t & 3t^{-1} \end{pmatrix}$$

Note that $\mathbf{u}(t)$ is the solution of the equation

$$\mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t) \iff \begin{pmatrix} t & t^{-1} \\ t & 3t^{-1} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2t^2 \\ 4t \end{pmatrix}.$$

We compute

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} t & t^{-1} \\ t & 3t^{-1} \end{pmatrix}^{-1} \begin{pmatrix} 2t^2 \\ 4t \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3t^{-1} & -t^{-1} \\ -t & t \end{pmatrix} \begin{pmatrix} 2t^2 \\ 4t \end{pmatrix}$$

and so

$$\begin{cases} u_1'(t) = 3t - 2 \\ u_2'(t) = -t^3 + 2t^2 \end{cases} \implies \begin{cases} u_1(t) = \frac{3}{2}t^2 - 2t \\ u_2(t) = -\frac{1}{4}t^4 + \frac{2}{3}t^3. \end{cases}$$

4. (15 pts) Consider the linear system

$$\mathbf{x}' = \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}}_{A} \mathbf{x}.$$

Then

$$\det(A - \lambda I) = -(\lambda - 3)^2(\lambda - 1) = 0,$$

and linearly independent eigenvectors are given as follows:

$$\left[\lambda = 3 \implies \boldsymbol{\xi} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right], \quad \left[\lambda = 1 \implies \boldsymbol{\xi} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]$$

Find the general solution of the system.

Solution) First solution corresponding to $\rho = 3$ is given by

$$\mathbf{x}^{(1)}(t) = e^{3t} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

Another solution corresponding to $\rho = 3$ is of the form

$$\mathbf{x}^{(2)}(t) = te^{3t}\boldsymbol{\xi} + e^{3t}\boldsymbol{\eta}$$

where η is a solution of

$$(A - 3I)\boldsymbol{\eta} = \boldsymbol{\xi} \iff \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

and so $-2\eta_1 + \eta_2 = 1$ and $\eta_3 = 2$. Set $\eta_1 = k$ so that $\eta_1 = 1 + 2k$ and

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} k \\ 1 + 2k \\ 2 \end{pmatrix} = k \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

We may take

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \implies \mathbf{x}^{(2)}(t) = te^{3t}\boldsymbol{\xi} + e^{3t}\boldsymbol{\eta} = te^{3t} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + e^{3t} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

A solution corresponding to $\rho = 1$ is given by

$$\mathbf{x}^{(3)}(t) = e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and so the general solution of the system is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + c_3\mathbf{x}^{(3)}(t) = c_1 e^{3t} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c_2 \left[te^{3t} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + e^{3t} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right] + c_3 e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

for some constants c_1, c_2, c_3.

5. Consider the initial value problem

$$\begin{cases} y' = \frac{2009y^2+4t^2}{3+3t^2}, \\ y(5) = 3. \end{cases}$$

The Runge-Kutta method is given as follows:

$$y_{n+1} = y_n + (h/6)(k_{n_1} + 2k_{n_2} + 2k_{n_3} + k_{n_4})$$

where

$$\begin{aligned} k_{n_1} &= f(t_n, y_n) \\ k_{n_2} &= f(t_n + (1/2)h, y_n + (1/2)hk_{n_1}) \\ k_{n_3} &= f(t_n + (1/2)h, y_n + (1/2)hk_{n_2}) \\ k_{n_4} &= f(t_n + h, y_n + hk_{n_3}). \end{aligned}$$

(a) (5 pts) What is the meaning of the global truncation error E_n?

(b) (5 pts) The general theory says that the Runge-Kutta method is a fourth order approximation method. What does "fourth order" mean?

(c) (5 pts) Choose the step size $h = 0.2$. Find n for which y_n approximates $y(5.8)$ when we use the Runge-Kutta method.

Solution) (a) Let $\phi(t)$ be the exact solution. The global truncation error $E_n = \phi(t_n) - y_n$ where y_n is the value obtained by the Runge-Kutta formula and y_{n-1} is also the (approximated) value obtained by the Runge-Kutta method for $n = 1, 2, \ldots$.

(b) The meaning of "fourth-order" is that $|E_n| \leq Ch^4$ where E_n is the global truncation error, $h = t_n - t_{n-1}$ is the step size and C is a constant.

(c) Note that

$$y(5) = 3 \implies t_0 = 5, \quad y_0 = 3.$$

Recall that $t_n = t_0 + nh$. Since we want to approximate $y(t)$ at $t = 5.8$ we find that

$$5.8 = t_n = 5 + n(0.2) \implies n = \frac{0.8}{0.2} = 4 \implies t_4 = 4 \text{ and } y_4 \approx y(t_4) = y(5.8).$$