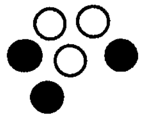
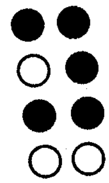
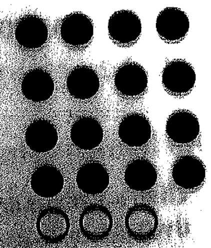
10.1 Ways of Thinking About Signed Numbers

1. What single integer is being represented by each chip display? (Use white for positive and black for negative, for consistency.)

a. 

b. 

c. 

2. "I don't get it! There are 8 chips, but you say that it shows 2." To what chip display is the student reacting, and what is missing in the student's understanding?

3. Put these numbers in order, from smallest to largest, without using decimals.

   $\frac{53}{6}$ $\quad {}^{-}(\frac{3}{4})$ $\quad {}^{-}(\frac{5}{4})$ $\quad \frac{53}{7}$ $\quad {}^{-}(\frac{53}{7})$ $\quad \frac{1}{1000}$ $\quad {}^{-}(\frac{1}{10{,}000})$ $\quad {}^{-}(\frac{53}{6})$

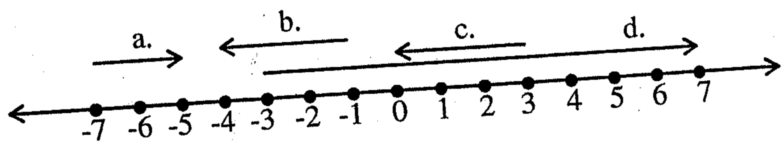
4. Why is the set of whole numbers *not* dense?

5. a. Give five rational numbers between ${}^{-}(\frac{3}{4})$ and ${}^{-}(\frac{1}{4})$.

   b. How many rational numbers are there, between ${}^{-}(\frac{3}{4})$ and ${}^{-}(\frac{1}{4})$?

6. Interpret this record of plays in a football game, in terms of yards gained or lost: $^{+}4, {}^{+}8, {}^{-}3, {}^{-}15, {}^{+}11$.

7. What single integer is represented by each arrow below?



10.2 Adding and Subtracting Signed Numbers

1. Illustrate $5 + {}^{-}2$ with each of the following.

   a. colored chips   b. a number line   c. a financial situation

2. a. Illustrate $5 - {}^{-}2$ with colored chips.

   b. Illustrate ${}^{-}5 - {}^{-}2$ with colored chips.

   c. Illustrate ${}^{-}5 - 2$ with colored chips.

3. a. Is the set of numbers, –1, 0, 1, closed under multiplication? Explain.

   b. Is the set of numbers, –1, 0, 1, closed under addition? Explain.

   c. Is the set of numbers, –2, 0, 2, closed under multiplication? Explain.

4. Calculate the following.

   a. $112 + {}^{-}78 + {}^{-}47 - 65$

   b. ${}^{-}563 - {}^{-}386 + {}^{-}291$

   c. ${}^{-}(\frac{3}{4}) + \frac{5}{12} - \frac{3}{8}$

   d. ${}^{-}4 + \frac{2}{3} - {}^{-}1\frac{1}{3} + 2\frac{1}{2}$

   e. ${}^{-}1.19 + 2.3 - {}^{-}0.6 + 0.003$

5. Use *only* the property given to change the given expression.
   a. $(7 + {}^{-}3) + (-15 + 8)$, commutative property of addition
   b. $(5 + {}^{-}5) + 0$, additive inverse property
   c. $(5 + {}^{-}5) + 0$, additive identity
   d. $[{}^{-}2 + ({}^{-}4 + 5)] + {}^{-}5$, associative property of addition

10.3 Multiplying and Dividing Signed Numbers

1. Give a pattern argument to suggest that ${}^{-}2 \times {}^{-}5 = {}^{+}10$, assuming that (negative) × (positive) = (negative) has been established.
2. Give the following products and quotients. (Can you do these without a calculator? You should be able to do so.)
   a. ${}^{-}57.3 \times {}^{-}9$ b. ${}^{-}(\frac{7}{24}) \times {}^{-}(\frac{15}{49}) \times \frac{14}{25}$ c. $7296 \div {}^{-}57$ d. ${}^{-}235.752 \div {}^{-}0.57$
3. Can your calculator multiply integers, or do you have to be aware of the sign for the product?
4. Use *only* the property given to change the given expression.
   a. $(\frac{5}{8} \times {}^{-}20) \times ({}^{-}4\frac{5}{6} \times \frac{8}{35})$, associativity of multiplication
   b. $(\frac{5}{8} \times {}^{-}20) \times ({}^{-}4\frac{5}{6} \times \frac{8}{35})$, commutativity of multiplication
   c. $\frac{5}{6} \times (24 + {}^{-}600)$, distributive property
   d. $(\frac{2}{3} \times {}^{-}298) + (\frac{2}{3} \times {}^{-}2)$, distributive property e. $2x + {}^{-}5x$, distributive property
   f. $(5 \times \frac{1}{5}) \times 1$, multiplicative inverses g. $(5 \times \frac{1}{5}) \times 1$, multiplicative identity
5. Use the chip model and repeated subtraction to make drawings to calculate the following.
   a. ${}^{-}10 \div {}^{-}2$ b. ${}^{-}16 \div {}^{-}2$

11.1 Factors and Multiples, Primes and Composites

1. Without looking the phrases up, tell what a prime number is, and what a composite number is.
2. Is it possible for some composite numbers to be prime numbers also?
3. T or F: An array for a prime number can have exactly one row or exactly one column. Explain your decision.
4. T or F: Except for 1, only composite numbers are crossed out in the sieve of Eratosthenes.
5. In the sieve of Eratosthenes, if 19 is the next number not crossed out, which one of the following will be the next new number to be crossed out? Explain.
   a. 23 b. 361 (= 19 × 19) c. 323 (= 19 × 17) d. 20
6. If *k* is a factor of *m*, is *m* always a factor of *k*? Explain.
7. Use an example to show how the words "multiple" and "factor" can be used for the same multiplication equation.
8. For each number below, write a sentence that refers to the number, for 87 × 24 = 2088.
   a. 87 b. 24 c. 2088

9. For each (whole number) variable below, write a sentence that refers to the variable, for $g \times s = w$.

a. g b. s c. w

10. a. Give three factors of 49. Can you find more? If so, how? If not, why not?

b. Is 49 a composite number? Explain why or why not.

c. Give three multiples of 49. Can you find more? If so, how? If not, why not?

11. Tell whether each of the following is a prime or composite number. Explain how you know.

a. 31 b. $13 \times 23 = 299$ c. 27 d. 999 e. $\frac{7}{11}$

12. Is the set of all prime numbers closed under multiplication? Explain.

13. a. List all of the multiples of 0. b. What whole numbers are factors of 0?

14. Find at least one example that shows the following conjecture is false:

The sum of two prime numbers is an even number.

11.2 Prime Factorization

1. Does the 10 in $1024 = 2^{10}$ keep 2^{10} from being the prime factorization of 1024? Explain.

2. What does the Unique Factorization Theorem say about $1024 = 2^{10}$?

3. Make a factor tree for each of the following, and give the prime factorization for each.

a. 960 b. 9600 c. 1125 d. 8100 e. 29

4. Kim and Lee start their factor trees for 5000 differently. Kim has branches to 5 and 1000, and Lee has branches to 50 and 100. Does this violate the Fundamental Theorem of Arithmetic? Explain.

5. What is contradictory in this statement: If k is an odd number and $k = 2m$, then k is a factor of m?

6. Why aren't the following results a contradiction to the Unique Factorization Theorem?

Abbie has $32{,}832 = 2^4 \times 2^2 \times 3^2 \times 57$ and Bonita has $32{,}832 = 2^6 \times 3 \times 171$.

7. Give (a) four prime factors and (b) six composite factors of 3,972,672 ($= 2^6 \times 3 \times 19 \times 33^2$).

8. Is there a whole number m such that $7^{15} = 9^m$? Explain.

9. For each part, find the smallest number with the given factors.

a. 14, 105, and 63 b. 33, 45, and 90 c. 98, 147, 35, and 420

10. a. T or F: Every nonzero multiple of 75 will have 3×5^2 in its prime factorization.

b. T or F: Every nonzero multiple of 24 will have $2^3 \times 3$ in its prime factorization.

c. Use parts (a) and (b) to find the smallest number that is a multiple of 75 and also a multiple of 24 (the least "common multiple").

11. Find, if possible, nonzero values for m and n such that $25^m = 125^n$. If it is not possible, explain why.

12. a. How many factors does 94,325 have? (Hint: $94{,}325 = 5^2 \times 7^3 \times 11$.)

b. If p, q, and r are different primes, what expression tells how many factors $p^x \times q^y \times r^z$ has?

13. Give the prime factorization of one million, and tell how many factors one million has.

14. How many factors does one billion have? How many of these are primes? How many are composites? (Hint: Don't forget 1.)

15. Give one number that has 21 as a factor and that has exactly 15 factors. Is there just one possibility?

11.3 Divisibility Tests to Determine Whether a Number Is Prime

1. Which pair(s) of numbers below are relatively prime? Explain.

   a. 27, 29 **b.** 25, 35 **c.** 25, 32 **d.** 100, 81 **e.** 125, 36

2. T or F: A divisibility test could be called a factor test instead.

3. Practice the divisibility tests for 2, 3, 4, 5, 6, 8, 9, and 10 on these numbers:

   a. 97,236 **b.** 1,000,000 **c.** 714,612 **d.** 7,000,005

   e. 2299 **f.** 494 **g.** 80×54 **h.** 400

4. In verifying that each of the following is a prime, what is the largest prime that must be checked as a possible factor?

   a. 401 **b.** 509 **c.** 887 **d.** 1607

5. Which of the following are primes? For those not prime, give the prime factorization.

   a. 207 **b.** 121 **c.** 5893 $(= 83 \times 71)$ **d.** 6859 $(= 19^3)$

   e. 247 **f.** 119 **g.** 97 **h.** 197 **i.** 297

6. Give one 12-digit number that has 3 as a factor, but not 9, and also 4 as a factor, but not 8.

7. Why is it not possible to give a 10-digit number that has 9 as a factor but does not have 3 as a factor?

11.4 Greatest Common Factor, Least Common Multiple

1. **a.** What is the GCF of 36 and 48? **b.** What is the LCM of 36 and 48?

   c. Simplify $\frac{36}{48}$ in one step. **d.** Add: $\frac{17}{36} + \frac{5}{48}$.

2. **a.** What is the GCF of 150 and 270? **b.** What is the LCM of 150 and 270?

   c. Simplify $\frac{150}{270}$ in one step. **d.** Add: $\frac{91}{150} + \frac{7}{270}$

3. For each of the following pairs of numbers, find the LCM and the GCF. You may leave your answers in factored form or multiply them out.

   a. 96 and 132 **b.** 42 and 126 **c.** 36 and 64

   d. $2^3 \times 7^2 \times 11$ and $2^2 \times 7^3 \times 13$ **e.** x^4y^6 and x^2y^9

4. Here is a method that is taught in another culture (Hong Kong), for finding the least common multiple of three numbers, like 12, 18, and 60 in the example.

First (3 is a common factor of all three)	Then	Finally (notice the 3 in 2 3 10 is just repeated)
3)12 18 60 4 6 20	3)12 18 60 2)4 6 20 2 3 10	3)12 18 60 2)4 6 20 2)2 3 10 1 3 5

   The divisions stop when no pair of numbers have a common factor. The LCM of 12, 18, and 60 is then obtained by multiplying the numbers on the outside: $3 \times 2 \times 2 \times 3 \times 5 = 180$. LCM(12, 18, 60) = 180.

   a. Try the algorithm to find the LCM of 24, 45, and 50. **b.** Why does this algorithm work?

3. From smallest to largest, $^-(\frac{53}{6})$ $^-(\frac{53}{7})$ $^-(\frac{5}{4})$ $^-(\frac{3}{4})$ $^-(\frac{1}{10{,}000})$ $\frac{1}{1000}$ $\frac{53}{7}$ $\frac{53}{6}$

4. There is no whole number between consecutive whole numbers, like 8 and 9.

5. **a.** Using equivalent fractions $^-(\frac{300}{400})$ and $^-(\frac{100}{400})$, it is easy to see several.

   b. Infinitely many.

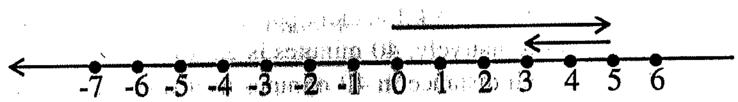
6. A gain of 4 yards, a gain of 8 yards, losses of 3 and 15 yards, a gain of 11 yards.

7. **a.** 2 or $^+2$ **b.** $^-3$ **c.** $^-3$ **d.** 10 or $^+10$

10.2 Adding and Subtracting Signed Numbers

1. **a.** 5 chips of one color for 5 and 2 chips of an "opposite" color for $^-2$, giving 3 chips of the color for positive after pairs of opposite colors cancel.

   b. (one way)



   c. Sample: In the mail, you received a rebate of \$5 but an overdue charge of \$2 for a bill you sent in late. How has your financial position changed?

2. **a.** Show 5 with 7 positive chips and 2 negative ones. Removing two negative chips gives 7.

   b. Show 5 negative chips and take away 2 of them. The remainder is $^-3$.

   c. Show $^-5$ with 7 negative chips and 2 positive chips. Take away the two positive chips, leaving $^-7$ as the answer.

3. **a.** Yes, the set, $^-1$, 0, 1, is closed under multiplication because the product of every choice of pairs of numbers from the set is also a number in the set.

   b. The set, $^-1$, 0, 1, is not closed under addition because, for example, $1 + 1 = 2$, and 2 is not in the set.

   c. The set of numbers, $^-2$, 0, 2, is not closed under multiplication because, for example, $2 \times 2 = 4$, and 4 is not in the set.

4. **a.** $^-78$ **b.** $^-468$ **c.** $^-(\frac{17}{24})$ **d.** $\frac{1}{2}$ **e.** 1.713

5. **a.** Samples: $(^-15 + 8) + (7 + {^-3})$; or $(^-3 + 7) + (^-15 + 8)$; or $(7 + {^-3}) + (8 + {^-15})$; or combinations of those three.

   b. $0 + 0$ **c.** $5 + {^-5}$

   d. Samples: $[(^-2 + {^-4}) + 5] + {^-5}$; or $^-2 + [(^-4 + 5) + {^-5}]$; or further uses of associativity on the expressions in square brackets. Notice that the order of the addends does not change.

10.3 Multiplying and Dividing Signed Numbers

1. Start with, say, $4 \times {^-5} = {^-20}$, $3 \times {^-5} = {^-15}$, $2 \times {^-5} = {^-10}$, etc., continuing to $^-2 \times {^-5}$, and look for a pattern.

2. **a.** 1088.7 **b.** $\frac{1}{20}$ **c.** $^-128$ **d.** 413.6

3. With simple calculators, the user must assign the sign to the product.

4. **a.** Possibles, all keeping the order of the factors the same: $\frac{5}{8} \times [{}^{-}20 \times ({}^{-}4\frac{5}{6} \times \frac{8}{35})]$; or $[(\frac{5}{8} \times {}^{-}20) \times {}^{-}4\frac{5}{6}] \times \frac{8}{35}$; or further applications to the expressions in square brackets.

   b. Change the order of the two factors within either set of parentheses, or change the order of the parenthetical expressions, $({}^{-}4\frac{5}{6} \times \frac{8}{35}) \times (\frac{5}{8} \times {}^{-}20)$

   c. $(\frac{5}{6} \times 24) + (\frac{5}{6} \times {}^{-}600)$ **d.** $\frac{2}{3} \times ({}^{-}298 + {}^{-}2)$ **e.** $(2 + {}^{-}5)x$ **f.** 1×1 **g.** $5 \times \frac{1}{5}$

5. **ab.** Show the dividend (the ⁻10 and the ⁻16), and repeatedly remove chips for the divisor (⁻2). The number of groups removed will be the quotient.

11.1 Factors and Multiples, Primes and Composites

1. A prime number is a whole number that has exactly two different factors. A composite number is a whole number greater than 1 that has more than two factors.

2. No, composite numbers by definition have more than two factors, and prime numbers are restricted to numbers with exactly two factors.

3. T A prime number p can be expressed only by $p \times 1$ or $1 \times p$.

4. T Each number crossed out is a multiple of some number greater than 1, so that number would be a third factor of the number crossed out.

5. b. 361. 23 won't be crossed out (it is the next prime), and 323 is already crossed out, when all the multiples of 17 were crossed out. 20 is also already crossed out, as a multiple of 2.

6. No. For example, 2 is a factor of 6, but 6 is not a factor of 2.

7. Many examples exist. One is, for 2 × 4 = 8, "8 is a multiple of 2" and "2 is a factor of 8" are both correct.

8. a. 87 is a factor of 2088. **b.** 24 is a factor of 2088.

   c. 2088 is a multiple of 87, or 2088 is a multiple of 24.

9. a. g is a factor of w. **b.** s is a factor of w.

   c. w is a multiple of g, or w is a multiple of s.

10. a. 1, 7, and 49 are the only factors of 49 because 49 is the square of 7.

   b. Yes, 49 is a composite number because it is larger than 1 and has more than two factors.

   c. 0, 49, 98, 147, 196, 245, . . . Multiply 49 by any whole number to get a multiple of 49.

11. a. 31 is a prime, because it has only 1 and 31 as factors.

   b. 299 is a composite, because 13 (or 23) is a third factor besides 1 and 299.

   c. 27 is a composite, because 3 (or 9) is a third factor besides 1 and 27.

   d. 999 is a composite, because 3 (and 333 or 111 or . . .) would be a third factor besides 1 and 999.

   e. *Prime* and *composite* refer only to whole numbers, not fractions different from whole numbers.

12. No, because the product of two primes would have more than two factors and thus not be a prime.

13. a. 0 is the only multiple of 0.

   b. Every whole number n is a factor of 0, because $n \times 0 = 0$.

14. Adding the prime number 2 to any other prime number will give a counterexample.

11.2 Prime Factorization

1. No. 2^{10} is just shorthand for $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$, which is a prime factorization.
2. The Unique Factorization Theorem says that *every* prime factorization of 1024 will involve exactly ten 2s as factors.
3. There are different (correct) factor trees for each part, except part e, which as a prime can have at most branches to 29 and 1, and usually branches that end in 1 are not used. Whatever your factor tree, you should have the following prime factorizations (with your factors possibly in a different order, of course).

   a. $960 = 2^6 \times 3 \times 5$ **b.** $9600 = 2^7 \times 3 \times 5^2$

   c. $1125 = 3^2 \times 5^3$ **d.** $8100 = 2^2 \times 3^4 \times 5^2$
4. No, these different starts do not violate the Fundamental Theorem of Arithmetic, because the theorem applies only to prime factorizations, and neither Kim nor Lee has a prime factorization yet.
5. If k is an odd number, it cannot have 2 as a factor, which is implied by $k = 2m$.
6. Neither Abbie nor Bonita has a complete prime factorization. $57 = 3 \times 19$, and $171 = 3^2 \times 19$, so with that further work, the complete prime factorizations would agree.
7. **a.** 2, 3, 11, and 19 are the only prime factors of 3,972,672.

   b. There are 164 possibilities! Some, besides 33, are 121, 57, 6, 12, 24, 209, . . . Just be certain that your composite factors involve only a selection of the prime factors in the prime factorization.
8. No. Unique factorization into primes assures that 7^{15} has only 7 as a prime factor, and that 9^m has only 3 as a prime factor. Thus the two can never be equal.
9. **a.** 630 (Did you realize that this is just the LCM?) **b.** 990 **c.** 2940
10. **a.** T A nonzero multiple of 75 will be $75n$ for some whole number n greater than 1, so the 3×5^2 must appear in the prime factorization of $75n$, by the unique factorization theorem.

    b. T, with reasoning like that in part (a).

    c. The smallest such number must be $2^3 \times 3 \times 5^2 = 600$.
11. Both sides involve only prime factors of 5, so it may be possible. $(5^2)^m = 5^{2m}$, and $(5^3)^n = 5^{3n}$, so any values of m and n that make $2m = 3n$ will be possible values. One example is $m = 3$ and $n = 2$. Any (nonzero) multiples of those will also work.
12. **a.** $(2+1)(3+1)(1+1) = 3(4)(2) = 24$ factors **b.** $(x+1)(y+1)(z+1)$ factors
13. One million $= 1{,}000{,}000 = 10^6 = (2 \times 5)^6 = 2^6 \times 5^6$, so one million has $7 \times 7 = 49$ factors.
14. 100, reasoning as in Exercise 12. Two are primes: 2 and 5. 1 is a factor. So the other 97 factors must be composites.
15. At least one 3 and at least one 7 must appear in the prime factorization of the number. Each could be raised to powers so that, from $15 = 3 \times 5$, one exponent is 2 and the other 4: $3^2 \times 7^4$, or $3^4 \times 7^2$, for example.

11.3 Divisibility Tests to Determine Whether a Number Is Prime

1. Relatively prime numbers have only 1 as a common factor. So the pairs in parts a, c, d, and e are relatively prime. In part b, the numbers have 5 as a common factor.

2. T

3. **a.** 2, 3, 4, 6, and 9 are divisors. **b.** 2, 4, 5, 8, and 10 are divisors.
   c. 2, 3, 4, and 6 are divisors. **d.** 3 and 5 are divisors.
   e. None of 2, 3, 4, 5, 6, 8, 9, 10 is a factor. **f.** Only 2 is a divisor.
   g. Each of 2, 3, 4, 5, 6, 8, 9, and 10. (Was it necessary to multiply 80 × 54 out?)
   h. 2, 4, 5, 8, and 10 are divisors.

4. With approximate square roots as the guide . . .
   a. 19 ($\sqrt{401} \approx 20$ and we need check only primes)
   b. 23 (29^2 would be too large) **c.** 29 **d.** 39

5. **a.** $207 = 3^2 \times 23$ **b.** $121 = 11^2$ **c.** 83×71 **d.** 19^3
   e. $247 = 13 \times 19$ **f.** $119 = 7 \times 17$ **g.** prime **h.** prime **i.** $297 = 3^3 \times 11$

6. Check that the sum of the digits in your number has 3 as a factor, but not 9, that the number named by the rightmost two digits has 4 as a factor, and that the number named by the rightmost three digits does not have 8 as a factor.

7. If 9 is a factor, then 3 is automatically a factor (because 9 = 3 × 3).

11.4 Greatest Common Factor, Least Common Multiple

1. **a.** 12 **b.** 144 **c.** Eliminate the GCF: $\frac{12\times3}{12\times4} = \frac{3}{4}$ **d.** $\frac{17}{36} + \frac{5}{48} = \frac{68}{144} + \frac{15}{144} = \frac{83}{144}$

2. **a.** 30 **b.** 1350 **c.** $\frac{150}{270} = \frac{30\times5}{30\times9} = \frac{5}{9}$ **d.** $\frac{91}{150} + \frac{7}{270} = \frac{819}{1350} + \frac{35}{1350} = \frac{854}{1350} \left(= \frac{427}{675}\right)$

3. **a.** LCM = 1056; GCF = 12
   b. LCM = 126; GCF = 42 (Did you see this one fast?)
   c. LCM = 576; GCF = 4
   d. LCM = $2^3 \times 7^3 \times 11 \times 13 = 392{,}392$; GCF = $2^2 \times 7^2 = 196$
   e. LCM = x^4y^9; GCF = x^2y^6

4. **a.** 1800
   b. The algorithm involves finding a common factor, then "removing" it by dividing, and continuing, removing other common factors. At the end, then, the common factors multiplied are the least ones whose product will be a common multiple of the original numbers.