MA 23200 Spring 2010

NAME:________________________________ Exam 3

PUID:________________________________

INSTRUCTIONS

- No books or notes are allowed.
- You may use a one-line scientific calculator. No other electronic device is allowed. Be sure to turn off your cellphone.
- Show all your work in the space provided. Little or no credit may be given for an answer with insufficient or inconsistent work, even if the answer happens to be correct.
- Write answers in the boxes provided. All answers are expected to be simplified ($\frac{2}{4} \to \frac{1}{2}$, $2x + x \to 3x$, $e^{\ln 2} \to 2$, etc).

Question	Possible	Score
1	5	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	15	

To approximate the solution to $y' = f(x, y)$, $y(x_0) = y_0$ using Euler's method with increments of Δx, we use the formula

$$y_{n+1} = y_n + f(x_n, y_n)\Delta x$$

where $y_n = y(x_n)$.

1.) (5 pts) Consider the initial value problem

$$y' + \frac{1}{x^2-1}y = \frac{\sin x}{2x-9} \qquad\qquad y(2) = 0.$$

What is the largest interval on which a *unique* continuous solution will exist. Do not attempt to find the solution.

$\frac{1}{x^2-1}$ is discontinuous at $x = 1$ or -1. $\frac{\sin x}{2x-9}$ is discontinuous at $x = 4.5$. The starting value is at $x_0 = 2$, which is between 1 and 4.5, so the largest interval on which a unique solution will exist is $(1, 4.5)$.

$(1, 4.5)$

2.) (10 pts) Find the general solution.

$$y' - 2y = 3x$$

$F(x) = \int -2dx = -2x$

$G(x) = e^{-2x}$

$$e^{-2x}(y' - 2y) = 3xe^{-2}$$

$$(e^{-2x}y)' = 3xe^{-2x}$$

$$e^{-2x}y = \int 3xe^{-2x}dx \qquad\qquad \begin{array}{ll} u = 3x & dv = e^{-2x}dx \\ du = 3dx & v = -\frac{1}{2}e^{-2x} \end{array}$$

$$e^{-2x}y = -\frac{3}{2}xe^{-2x} + \int \frac{3}{2}e^{-2x}dx$$

$$e^{-2x}y = -\frac{3}{2}xe^{-2x} - \frac{3}{4}e^{-2x} + C$$

$$y = -\frac{3}{2}x - \frac{3}{4} + Ce^{2x}$$

$y = y = -\frac{3}{2}x - \frac{3}{4} + Ce^{2x}$

3.) (10 pts) Find y.

$$xy' + 4y = x^5 \qquad\qquad y(1) = 2$$

$$y' + \frac{4}{x}y = x^4$$

$F(x) = \int \frac{4}{x}dx = 4\ln|x| = \ln x^4 \qquad G(x) = e^{\ln x^4} = x^4.$

$$x^4\left(y' + \frac{4}{x}y\right) = x^8$$

$$(x^4y)' = x^8$$

$$x^4y = \frac{1}{9}x^9 + C$$

$y(1) = 2 \Rightarrow 2 = \frac{1}{9} + C \Rightarrow C = \frac{17}{9}$

$$x^4y = \frac{1}{9}x^9 + \frac{17}{9}$$

$$y = \frac{1}{9}x^5 + \frac{17}{9x^4}$$

$y = y = \frac{1}{9}x^5 + \frac{17}{9x^4}$

4.) (10 pts) Find the general solution, solving completely for y.

$$\frac{dy}{dx} = 5x^4(2y-5)^4$$

$$\int \frac{1}{(2y-5)^4}dy = \int 5x^4dx \Rightarrow -\frac{1}{6(2y-5)^3} = x^5 + C$$

$$(2y-5)^3 = \frac{1}{C-6x^5} \Rightarrow 2y-5 = \sqrt[3]{\frac{1}{C-6x^5}} \Rightarrow y = \frac{\sqrt[3]{\frac{1}{C-6x^5}} + 5}{2}$$

$y = \frac{\sqrt[3]{\frac{1}{C-6x^5}}+5}{2}$

5.) (10 pts) Solve completely for y.

$$y' = y + \frac{1}{y} \qquad\qquad y(0) = -7$$

$$\int \frac{1}{y + \frac{1}{y}} dy = \int dx \Rightarrow \int \frac{y}{y^2+1} dy = \int dx \Rightarrow \frac{1}{2}\ln(y^2+1) = x + C$$

$y(0) = -7 \Rightarrow \frac{1}{2}\ln 50 = C$

$$\frac{1}{2}\ln(y^2+1) = x + \frac{1}{2}\ln 50 \Rightarrow \ln(y^2+1) = 2x + \ln 50 \Rightarrow y^2+1 = 50e^{2x} \Rightarrow y = -\sqrt{50e^{2x}-1}$$

$y = -\sqrt{50e^{2x}-1}$

6.) (10 pts) Find the equilibrium solutions of the autonomous differential equation

$$y' = 3y^2 - 5y^3$$

and classify each one as either **asymptotically stable**, **unstable** or **semistable**.

$3y^2 - 5y^3 = 0 \Rightarrow y^2(3-5y) = 0 \Rightarrow y = 0, \frac{3}{5}$

$y > \frac{3}{5}$	$-$	$\searrow$
$y = \frac{3}{5}$	0	$\rightarrow$
$0 < y < \frac{3}{5}$	$+$	$\nearrow$
$y = 0$	0	$\rightarrow$
$y < 0$	$+$	$\nearrow$

$y = \frac{3}{5}$ asymptotically stable
$y = 0$ semi-stable

7.) (10 pts) Let y be a solution to the initial value problem

$$y' = x(y^2 + 1) \qquad\qquad y(1) = 2.$$

Use Euler's Method (see page 1) with $\Delta x = 0.5$ to approximate $y(2)$. Round your answer to at least four decimal places.

$$y(1) = 2$$
$$y(1.5) = 2 + (5)(0.5) = 4.5$$
$$y(2) = 4.5 + 1.5(21.25)(0.5) = 20.4375$$

$y(2) \approx 20.4375$

8.) Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -2 \\ 3 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 0 & 4 \\ 1 & -1 & 3 \\ 0 & 2 & -1 \end{bmatrix}$$

Compute the following:

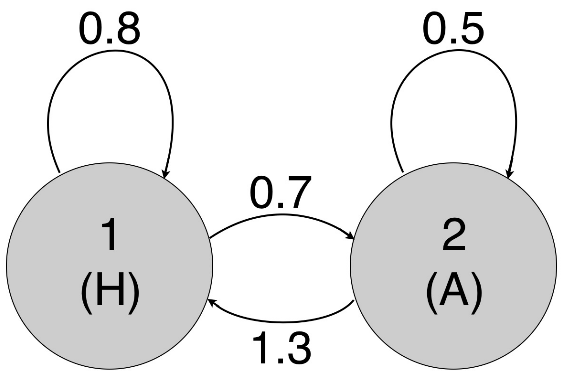
a.) (5 pts) $\mathbf{A} - \mathbf{B}$

$$\begin{bmatrix} -1 & 2 & -6 \\ 2 & 0 & -3 \\ -2 & -1 & 2 \end{bmatrix}$$

b.) (5 pts) $\mathbf{AB}$

$$\begin{bmatrix} 4 & -6 & 12 \\ 5 & 1 & 9 \\ -3 & 1 & -6 \end{bmatrix}$$

9.) The population of a certain species of birds is divided into two groups: hatchlings (H) and adults (A). The Leslie diagram associated to this species is given below.



a.) (5 pts) Give the Leslie Matrix for this bird population.

$$\begin{bmatrix} 0.8 & 1.3 \\ 0.7 & 0.5 \end{bmatrix}$$

b.) (5 pts) If there are 300 hatchlings and 150 adults in year 1, how many hatchlings and how many adults will there be in year 2?

$$\begin{bmatrix} 0.8 & 1.3 \\ 0.7 & 0.5 \end{bmatrix} \begin{bmatrix} 300 \\ 150 \end{bmatrix} = \begin{bmatrix} 435 \\ 285 \end{bmatrix}$$

Hatchlings:435

Adults:285

10.) (15 pts) Solve the following system of linear equations using augmented matrices and Gaussian elimination or Gauss-Jordan Elimination. No credit will be given for any other method.

$$\begin{array}{rcrcrcr} x & + & 7y & - & 3z & = & -8 \\ 3x & - & y & + & 3z & = & 15 \\ & & y & + & 5z & = & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 7 & -3 & -8 \\ 3 & -1 & 3 & 15 \\ 0 & 1 & 5 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 7 & -3 & -8 \\ 0 & -22 & 12 & 39 \\ 0 & 1 & 5 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 7 & -3 & -8 \\ 0 & 1 & 5 & 1 \\ 0 & -22 & 12 & 39 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -38 & -15 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 122 & 61 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -38 & -15 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

$(x, y, z) = \left(4, -\frac{3}{2}, \frac{1}{2}\right)$