1. Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

(a) $\vec{v} = \langle a, b, c \rangle = ai + bj + ck$; vector addition and subtraction geometrically using parallelograms spanned by $\vec{u}$ and $\vec{v}$; length or magnitude of $\vec{v} = \langle a, b, c \rangle$, $|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$; directed vector from $P_0(x_0, y_0, z_0)$ to $P_1(x_1, y_1, z_1)$ given by $\vec{v} = \overrightarrow{P_0P_1} = P_1 - P_0 = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$.

(b) Dot (or inner) product of $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$: $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$; properties of dot product; useful identity: $\vec{a} \cdot \vec{a} = |\vec{a}|^2$; angle between two vectors $\vec{a}$ and $\vec{b}$: $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$; $\vec{a} \perp \vec{b}$ if and only if $\vec{a} \cdot \vec{b} = 0$; the vector in $\mathbb{R}^2$ with length $r$ with angle $\theta$ is $\vec{v} = \langle r \cos \theta, r \sin \theta \rangle$.

(c) Projection of $\vec{b}$ along $\vec{a}$: $\text{proj}_a \vec{b} = \left\{ \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \right\} \frac{\vec{a}}{|\vec{a}|}$; Work $= \vec{F} \cdot \vec{D}$.

(d) Cross product (only for vectors in $\mathbb{R}^3$):

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

properties of cross products; $\vec{a} \times \vec{b}$ is perpendicular (orthogonal or normal) to both $\vec{a}$ and $\vec{b}$; area of parallelogram spanned by $\vec{a}$ and $\vec{b}$ is $A = |\vec{a} \times \vec{b}|$.

the area of the triangle spanned is $A = \frac{1}{2} |\vec{a} \times \vec{b}|$.
Volume of the parallelepiped spanned by $\vec{a}, \vec{b}, \vec{c}$ is $V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$

2. Equation of a line $L$ through $P_0(x_0, y_0, z_0)$ with direction vector $\vec{d} = \langle a, b, c \rangle$:

**Vector Form:** $\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t \vec{d}$.

**Parametric Form:**

\[
\begin{align*}
  x &= x_0 + at \\
y &= y_0 + bt \\
z &= z_0 + ct
\end{align*}
\]

**Symmetric Form:**

\[
\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \quad \text{(If say $b = 0$, then $\frac{x - x_0}{a} = \frac{z - z_0}{c}$, $y = y_0$.)}
\]

3. Equation of the plane through the point $P_0(x_0, y_0, z_0)$ and perpendicular to the vector $\vec{n} = \langle a, b, c \rangle$ ($\vec{n}$ is a normal vector to the plane) is $\langle (x - x_0), (y - y_0), (z - z_0) \rangle \cdot \vec{n} = 0$; Sketching planes (consider $x, y, z$ intercepts).

4. Quadric surfaces (can sketch them by considering various *traces*, i.e., curves resulting from the intersection of the surface with planes $x = k, y = k$ and/or $z = k$); some generic equations have the form:

(a) **Ellipsoid:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

(b) **Elliptic Paraboloid:** $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

(c) **Hyperbolic Paraboloid (Saddle):** $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

(d) **Cone:** $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

(e) **Hyperboloid of One Sheet:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

(f) **Hyperboloid of Two Sheets:** $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
5. Vector-valued functions $\mathbf{r}(t) = (f(t), g(t), h(t))$; tangent vector $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, unit normal vector $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$ differentiation rules for vector functions, including:

(i) $\{\phi(t) \mathbf{v}(t)\}' = \phi(t) \mathbf{v}'(t) + \phi'(t) \mathbf{v}(t)$, where $\phi(t)$ is a real-valued function

(ii) $(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u} \cdot \mathbf{v}' + \mathbf{u}' \cdot \mathbf{v}$

(iii) $(\mathbf{u} \times \mathbf{v})' = \mathbf{u} \times \mathbf{v}' + \mathbf{u}' \times \mathbf{v}$

(iv) $\{\mathbf{v}(\phi(t))\}' = \phi'(t) \mathbf{v}'(\phi(t))$, where $\phi(t)$ is a real-valued function

6. Integrals of vector functions \[ \int \mathbf{r}(t) \, dt = \left\langle \int f(t) \, dt, \int g(t) \, dt, \int h(t) \, dt \right\rangle; \] arc length of curve parameterized by $\mathbf{r}(t)$ is $L = \int_{a}^{b} |\mathbf{r}'(t)| \, dt$; arc length function $s(t) = \int_{a}^{t} |\mathbf{r}'(u)| \, du$; reparameterize by arc length: $\mathbf{\tilde{r}}(s) = \mathbf{r}(t(s))$, where $t(s)$ is the inverse of the arc length function $s(t)$; the curvature of a curve parameterized by $\mathbf{r}(t)$ is $\kappa = |\mathbf{T}'(t)| = |\mathbf{T}'(t)|$. \textbf{Note:} $\sqrt{\alpha^2} = |\alpha|$.

7. $\mathbf{r}(t) = $ position of a particle, $\mathbf{r}'(t) = \mathbf{v}(t) =$ velocity; $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) =$ acceleration; $|\mathbf{r}'(t)| = |\mathbf{v}(t)| =$ speed; Newton’s 2\textsuperscript{nd} Law: $\mathbf{F} = m \mathbf{a}$.

8. Domain and range of a function $f(x, y)$ and $f(x, y, z)$; level curves (or contour curves) of $f(x, y)$ are the curves $f(x, y) = k$; using level curves to sketch surfaces; level surfaces of $f(x, y, z)$ are the surfaces $f(x, y, z) = k$.

9. Limits of functions $f(x, y)$ and $f(x, y, z)$; limit of $f(x, y)$ does not exist if different approaches to $(a, b)$ yield different limits; continuity.

10. Partial derivatives $\frac{\partial f}{\partial x}(x, y) = f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$,

$\frac{\partial f}{\partial y}(x, y) = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$; higher order derivatives: $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$,

$f_{yy} = \frac{\partial^2 f}{\partial y^2}$, $f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$, etc; mixed partials.

11. Equation of the tangent plane to the graph of $z = f(x, y)$ at $(x_0, y_0, z_0)$ is given by

$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

12. Total differential for $z = f(x, y)$ is $dz = df = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy$; total differential for $w = f(x, y, z)$ is $dw = df = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy + \frac{\partial f}{\partial z} \, dz$; linear approximation for $z = f(x, y)$ is given by $\Delta z \approx dz$, i.e., $f(x + \Delta x, y + \Delta y) - f(x, y) \approx \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy$, where $\Delta x = dx$, $\Delta y = dy$;

Linearization of $f(x, y)$ at $(a, b)$ is given by $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$; $L(x, y) \approx f(x, y)$ near $(a, b)$.
13. Different forms of the \textbf{Chain Rule}: Form 1, Form 2; General Form: Tree diagrams. For example:

(a) If \( z = f(x, y) \) and \( \begin{align*}
   x &= x(t) \\
   y &= y(t)
\end{align*} \), then
\[
   \frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.
\]

(b) If \( z = f(x, y) \) and \( \begin{align*}
   x &= x(s, t) \\
   y &= y(s, t)
\end{align*} \), then
\[
   \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}.
\]

etc.....

14. \textbf{Implicit Differentiation}:

\textbf{Part I}: If \( F(x, y) = 0 \) defines \( y \) as function of \( x \) (i.e., \( y = y(x) \)), then to compute \( \frac{dy}{dx} \), differentiate both sides of the equation \( F(x, y) = 0 \) w.r.t. \( x \) and solve for \( \frac{dy}{dx} \).

If \( F(x, y, z) = 0 \) defines \( z \) as function of \( x \) and \( y \) (i.e. \( z = z(x, y) \) ), then to compute \( \frac{\partial z}{\partial x} \), differentiate the equation \( F(x, y, z) = 0 \) w.r.t. \( x \) (hold \( y \) fixed) and solve for \( \frac{\partial z}{\partial x} \). For \( \frac{\partial z}{\partial y} \), differentiate the equation \( F(x, y, z) = 0 \) w.r.t. \( y \) (hold \( x \) fixed) and solve for \( \frac{\partial z}{\partial y} \).

\textbf{Part II}: If \( F(x, y) = 0 \) defines \( y \) as function of \( x \) \( \iff \) \( \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \);

while if \( F(x, y, z) = 0 \) defines \( z \) as function of \( x \) and \( y \) \( \iff \) \( \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \) and \( \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} \).
15. Gradient vector for $f(x,y)$: $\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$, properties of gradients; gradient points in direction of maximum rate of increase of $f$; $\nabla f(x_0,y_0) \perp$ level curve $f(x,y) = C$ and, in the case of 3 variables, $\nabla f(x_0,y_0,z_0) \perp$ level surface $f(x,y,z) = C$:

16. Directional derivative of $f(x,y)$ at $(x_0,y_0)$ in the direction $\vec{u}$: $D_{\vec{u}} f(x_0,y_0) = \nabla f(x_0,y_0) \cdot \vec{u}$, where $\vec{u}$ must be a unit vector; tangent planes to level surfaces $f(x,y,z) = C$ (a normal vector at $(x_0,y_0,z_0)$ is $\vec{n} = \nabla f(x_0,y_0,z_0)$).