Study Guide # 2

1. Relative/local extrema; critical points ($\nabla f = \vec{\mathbf{0}}$ or ∇f does not exist); 2^{nd} Derivatives Test: A critical points is a local min if $D = f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$, local max if $D > 0$ and $f_{xx} < 0$, saddle if $D < 0$; absolute extrema; Max-Min Problems; **Lagrange Multipliers:** *Extremize* $f(\vec{\mathbf{x}})$ subject to a constraint $g(\vec{\mathbf{x}}) = C$, solve the system: $\nabla f = \lambda \nabla g$ and $g(\vec{\mathbf{x}}) = C$.

2. Double integrals; Midpoint Rule for rectangle : $\displaystyle\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x_i}, \overline{y_j})\,\Delta A$;

3. Type I region $D : \begin{cases} g_1(x) \le y \le g_2(x) \\ a \le x \le b \end{cases}$; Type II region $D : \begin{cases} h_1(y) \le x \le h_2(y) \\ c \le y \le d \end{cases}$;

iterated integrals over Type I and II regions: $\displaystyle\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$ and $\displaystyle\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$, respectively; Reversing Order of Integration (regions that are both Type I and Type II); properties of double integrals.

4. Integral inequalities: $mA \le \displaystyle\iint_D f(x,y)\,dA \le MA$, where $A =$ area of D and $m \le f(x,y) \le M$ on D.

5. Change of Variables Formula in Polar Coordinates: if $D : \begin{cases} h_1(\theta) \le r \le h_2(\theta) \\ \alpha \le \theta \le \beta \end{cases}$, then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\ r\sin\theta)\,\underset{\uparrow}{r}\,dr\,d\theta.$$

6. Applications of double integrals:

(a) Area of region D is $A(D) = \displaystyle\iint_D dA$

(b) Volume of solid under graph of $z = f(x,y)$, where $f(x,y) \ge 0$, is $V = \displaystyle\iint_D f(x,y)\,dA$

(c) Mass of D is $m = \displaystyle\iint_D \rho(x,y)\,dA$, where $\rho(x,y) =$ density (per unit area); sometimes write $m = \displaystyle\iint_D dm$, where $dm = \rho(x,y)\,dA$.

(d) Moment about the x-axis $M_x = \displaystyle\iint_D y\,\rho(x,y)\,dA$; moment about the y-axis $M_y = \displaystyle\iint_D x\,\rho(x,y)\,dA$.

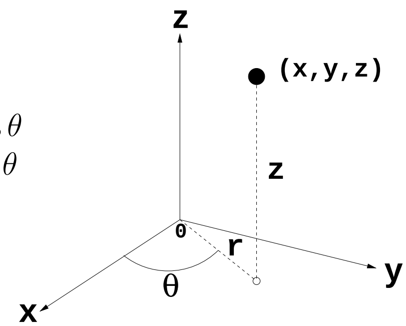
(e) Center of mass $(\overline{x}, \overline{y})$, where $\overline{x} = \dfrac{M_y}{m} = \dfrac{\iint_D x\,\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA}$, $\overline{y} = \dfrac{M_x}{m} = \dfrac{\iint_D y\,\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA}$

<u>Remark</u>: centroid = center of mass when density is constant (this is useful).

7. Elementary solids $E \subset \mathbb{R}^3$ of Type 1, Type 2, Type 3; triple integrals over solids E:
$\iiint_E f(x,y,z)\,dV = \iint_D \int_{u(x,y)}^{v(x,y)} f(x,y,z)\,dz\,dA$ for $E = \{(x,y) \in D,\ u(x,y) \le z \le v(x,y)\}$;
volume of solid E is $V(E) = \iiint_E dV$; applications of triple integrals, mass of a solid, moments about the coordinate planes M_{xy}, M_{xz}, M_{yz}, center of mass of a solid $(\overline{x}, \overline{y}, \overline{z})$.

8. Cylindrical Coordinates (r, θ, z):

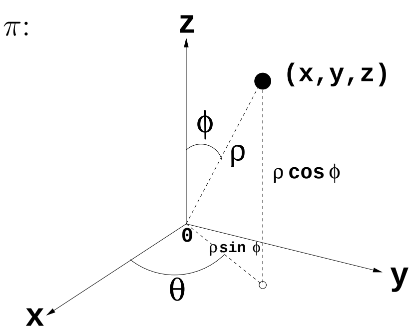
From CC to RC : $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$



Going from RC to CC use $x^2 + y^2 = r^2$ and $\tan\theta = \dfrac{y}{x}$ (make sure θ is in correct quadrant).

9. Spherical Coordinates (ρ, θ, ϕ), where $0 \le \phi \le \pi$:

From SC to RC : $\begin{cases} x = (\rho\sin\phi)\cos\theta \\ y = (\rho\sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases}$



Going from RC to SC use $x^2 + y^2 + z^2 = \rho^2$, $\tan\theta = \dfrac{y}{x}$ and $\cos\phi = \dfrac{z}{\rho}$.

10. Triple integrals in Cylindrical Coordinates: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$, $\boldsymbol{dV = r\,dz\,dr\,d\theta}$

$$\iiint_E f(x,y,z)\ dV = \iiint_E f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$
$$\uparrow$$

11. Triple integrals in Spherical Coordinates: $\begin{cases} x = (\rho\sin\phi)\cos\theta \\ y = (\rho\sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases}$, $\boldsymbol{dV = \rho^2 \sin\phi\ \ d\rho\,d\phi\,d\theta}$

$$\iiint_E f(x,y,z)\ dV = \iiint_E f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$\uparrow$$

12. Vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$: $\vec{\mathbf{F}}(x,y) = \langle P(x,y),\ Q(x,y)\rangle$ and $\vec{\mathbf{F}}(x,y,z) = \langle P(x,y),\ Q(x,y),\ R(x,y)\rangle$;
$\vec{\mathbf{F}}$ is a conservative vector field if $\vec{\mathbf{F}} = \nabla f$, for some real-valued function f.

13. Line integral of a function $f(x,y)$ along C, parameterized by $x = x(t), \quad y = y(t)$ and $a \le t \le b$, is

$$\int_C f(x,y)\; ds = \int_a^b f(x(t),\, y(t))\, \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt\,.$$

(independent of orientation of C, other properties and applications of line integrals of f)

<u>**Remarks**</u>:

(a) $\int_C f(x,y)\, ds$ is sometimes called the *"line integral of f with respect to arc length"*

(b) $\int_C f(x,y)\, dx = \int_a^b f(x(t),y(t))\, x'(t)\, dt$

(c) $\int_C f(x,y)\, dy = \int_a^b f(x(t),y(t))\, y'(t)\, dt$

14. Line integral of vector field $\vec{\mathbf{F}}(x,y)$ along C, parameterized by $\vec{\mathbf{r}}(t)$ and $a \le t \le b$, is given by

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt\,.$$

(depends on orientation of C, other properties and applications of line integrals of f)

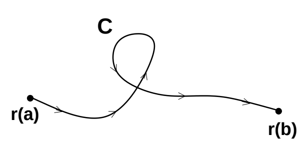
15. Connection between line integral of vector fields and line integral of functions:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C (\vec{\mathbf{F}} \cdot \vec{\mathbf{T}})\, ds$$

where $\vec{\mathbf{T}}$ is the unit tangent vector to the curve C.

16. If $\vec{\mathbf{F}}(x,y) = P(x,y)\,\vec{\mathbf{i}} + Q(x,y)\,\vec{\mathbf{j}}$, then $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C P(x,y)\, dx + Q(x,y)\, dy$; Work $= \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

17. <u>Fundamental Theorem of Calculus for Line Integrals</u>: $\int_C \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$:



18. A vector field $\vec{\mathbf{F}}(x,y) = P(x,y)\,\vec{\mathbf{i}} + Q(x,y)\,\vec{\mathbf{j}}$ is *conservative* (i.e. $\vec{\mathbf{F}} = \nabla f$) if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$; how to determine a potential function f if $\vec{\mathbf{F}}(\vec{\mathbf{x}}) = \nabla f(\vec{\mathbf{x}})$.

19. <u>Green's Theorem</u>: $\int_C P(x,y)\, dx + Q(x,y)\, dy = \iint_D \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\, dA$ ($C =$ boundary of D):