Sept. 16 2010 **MA 525** Solutions

(10) 1. Let $f(z) = x^2 + iy^2$. For what $z = x + iy$ does $f'(z)$ exist? Where is f analytic?

Cauchy-Riemann $u = x^2$ $v = y^2$

$u_x = v_y$ $u_y = -v_x$

$2x = 2y$! $x = y$ $0 = 0$

So $f'(z)$ exists on line $y = x$. But f is never analytic since to be analytic at z_0 f' has to exist in a neighborhood of z_0

(20) 2. Prove using the definitions introduced in this course that

$$\sin z_1 - \sin z_2 = 2\cos(\frac{z_1 + z_2}{2})\sin(\frac{z_2 - z_1}{2}).$$

Do algebra with right side!

$$2\cos\left(\frac{z_1+z_2}{2}\right)\sin\left(\frac{z_2-z_1}{2}\right)$$

$$= 2\,\frac{e^{i\left(\frac{z_1+z_2}{2}\right)} - e^{-i\left(\frac{z_1+z_2}{2}\right)}}{2} \cdot \frac{e^{i\left(\frac{z_2-z_1}{2}\right)} - e^{-i\left(\frac{z_2-z_1}{2}\right)}}{2i}$$

(20) 3. (a) Write the $z = \sin w$ and then invert it to derive an equation for the function $w = \sin^{-1} z$ (this will involve logarithms).

two-valued

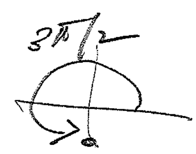
$$\frac{e^{iw}-e^{-iw}}{2i} = z, \quad e^{2iw} - 2iz e^{iw} - 1 = 0$$

Quadratic formula! $e^{iw} = \dfrac{2iz + \sqrt{4-4z^2}}{2} = iz + \sqrt{1-z^2}$

$$\therefore \; w = \sin^{-1} z = \frac{1}{i}\log\left(iz + \sqrt{1-z^2}\right)$$

(b) Use the formula from the first part, using what you know about the logarithm, to find all solutions to the equation

$$\sin^{-1}(-1).$$



Let $z = -1$

$$w = -i \log(-i + 0) = -i\left[\log 1 + i \arg -i\right]$$

$$= -i\left[i\left(\frac{3\pi}{2} + 2k\pi\right)\right] = \frac{3\pi}{2} + 2\pi k$$

$$k = 0, \pm 1, \pm 2, \ldots$$

(15) 4. Use the Cauchy-Riemann equations to find all harmonic conjugates to the harmonic function $u(z) = e^{2x}\cos 2y + xy$. (You make take for granted that u is harmonic.) *You must show work.*

(1) $u_x = v_y$, $u_y = -v_x$

By (1) $u_x = v_y = 2e^{2x}\cos 2y + y$

integrate w.r. to y:

$$v = e^{2x}\sin 2y + \frac{1}{2}y^2 + h(x)$$

Now diff w.r. to x

$$v_x = 2e^{2x}\sin 2y + h'(x) \overset{?}{=} -u_y \overset{!}{=} 2e^{2x}\sin 2y - x$$

$$\therefore h'(x) = -x, \quad h(x) = -\frac{1}{2}x^2 + C$$

$$v(x,y) = e^{2x}\sin 2y + \frac{1}{2}y^2 - \frac{1}{2}x^2 + C$$

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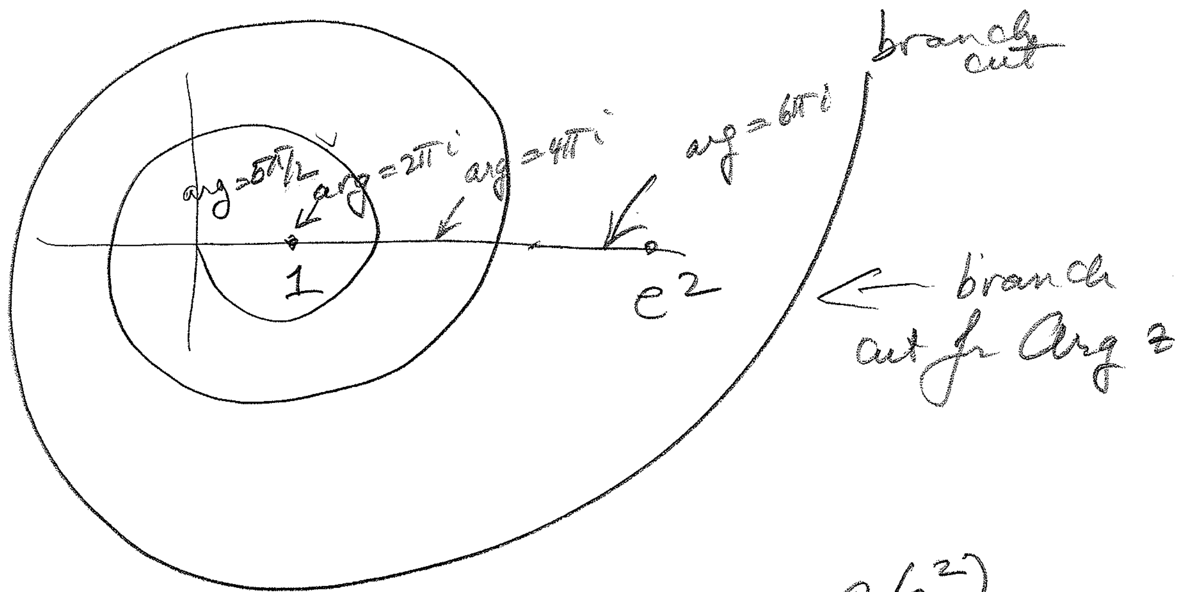
(15) 5. Describe how you could find a (single-valued) branch $\mathcal{L}og z$ of the multiple-valued function $w = \log z$ so that

$$\mathcal{L}og\,(1) = 2\pi i; \mathcal{L}og\,(e^2) = 2 + 6\pi i.$$

Convince me that you know how to chose such a branch (a good illustration is enough).



$$\mathcal{L}og\,(e^2) = \log e^2 + i\,\arg(e^2)$$
$$= 2 + 6\pi i$$