Oct 21, 2010 MA 525 'USE PENCIL! Solutions

(25) 1. True-false. If the statement is (always) true, indicate a reason, if it is false, explain why with a *counterexample* or a reason.

(*a*) If $f(z)$ is analytic in $\{|\arg z| < \pi\}$, then there is an analytic function F with $F' = f$. True. We need that if R is any rectangle in Ω, then $\int_{\partial R} f\,dz = 0$. But f will be analytic inside and on the boundary of R.

(*b*) If u is harmonic in a domain Ω, there is an analytic function f with $u = \Re(f)$.

False, $\Omega = \{1 < |z| < 2\}$, $\log|z|$ is harmonic but has no conjugate

(*c*) The function $u(z) = \log|z|$ harmonic for $\{1 < |z| < 2\}$.

Yes, At any such z_0, u is the real part of an analytic branch of $\log z$ near z_0

(*d*) Let α be a complex number. Then $1^\alpha = 1$.

False $1^\alpha = e^{\alpha \log 1} = e^{\alpha(2\pi k i)}$, Let $k=1, \alpha = i$ Then $1^\alpha = e^{-2\pi}$

(*e*) Let f be analytic at every point on the closed curve γ. Then $\int_\gamma f(z)\,dz = 0$.

False Ω 1 $f(z) = \frac{1}{z}$. But if Ω is not simply connected and f has a pole in the "hole", $\int f$ might be zero anyway! Ex! $f(z) = \frac{1}{z^4}$

(15) 2. Let $\Omega = \{1 \leq |z| \leq 2\}$. Find a harmonic function $u(z)$ with $u(z) = 5$ on $\{|z| = 1\}$ and $u(z) = -3$ for $\{|z| = 2\}$. Include a drawing.

Try $u(z) = A \log|z| + B$

$|z| = 1 \quad 5 = B$

$|z| = 2 \quad -3 = A \log 2 + 5 \quad ; \quad A = -\frac{8}{\log 2}$

$u(z) = -\frac{8}{\log 2}|z| + 5$

(20) 3. (*a*) Carefully define (using the logarithm) the function z^π, with $z = x + iy = re^{i\theta}$.

$z^\pi = e^{\pi \log z}$ $\left(= e^{\pi(\log|z| + i\arg z)} = e^{\pi(\log|z| + i\arg z + 2k\pi)}\right.$ (k an integer)$\left.\right)$

(*b*) Use this definition to show that $(d/dz)z^\pi = \pi z^{\pi-1}$.

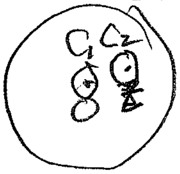
$$\frac{d}{dz} z^\pi = \frac{d}{dz} e^{\pi \log z} = e^{\pi \log z} \cdot \frac{\pi}{z} = z^\pi \cdot \frac{\pi}{z} = \pi z^{\pi - 1}$$

(40) 4. Short computations.

(*a*) Find $\displaystyle\int_{\{|z|=1\}} \frac{\cos z}{z^6}\, dz = \frac{2\pi i \frac{d^{(5)}}{dz^5}\cos(z)\Big|_{z=0}}{5!}$

But that is $\displaystyle\frac{2\pi i \cdot \sin 0}{5!} = 0$

(*b*) Compute the analytic function (we showed in class it is analytic!) $g(z) = \displaystyle\int_{\{|\zeta|=1\}} \frac{\bar{\zeta}^2}{\zeta - z}\, d\zeta$ for $\{|z| < 1\}$. We did several of these in class. $\bar\zeta^2$ is not analytic, so can't use Cauchy. But when $|\zeta| = 1$, $\bar\zeta = \frac{1}{\zeta}$, so this is



$\displaystyle\int \frac{1}{\zeta^2} \cdot \frac{1}{\zeta - z}\, d\zeta$. Let's compute as $\displaystyle\int_{C_1} + \int_{C_2}$ (so $z \neq 0$).

Then $\displaystyle\int_{C_1} + \int_{C_2} = 2\pi i \left[\frac{d^2}{d\zeta^2}\left(\frac{1}{\zeta - z}\right)_{\zeta=0} + \frac{1}{z^2}\right]$

$$= 2\pi i \left[-\frac{1}{(\zeta - z)^2}\Big|_{\zeta=0} + \frac{1}{z^2}\right] = 2\pi i \left[-\frac{1}{z^2} + \frac{1}{z^2}\right] = 0$$

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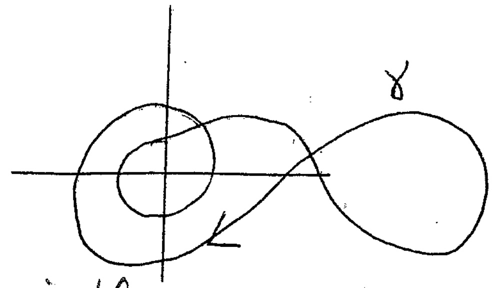
Can compute for $z = 0$, but $g \equiv 0$ for $z \neq 0$, and g is analytic, so $g(0) = 0$.

(c) $\int_\gamma \frac{1}{z}\,dz$ with γ as drawn



"γ 'goes around 0' twice in the negative direction". Ans is $-4\pi i$

(d) $\int_{\{|z|=2\}} \frac{\sin z}{z^2(z-4)}\,dz.$ $= 2\pi i\left[\frac{d}{dz}\left(\frac{\sin z}{z-4}\right)\Big|_{z=0}\right]$

$= 2\pi i \left[\frac{(z-4)\cdot\cos z - \sin z}{(z-4)^2}\right]\Big|_{z=0} = \frac{2\pi i}{-4} = -\frac{\pi i}{2}$

(e) $\int_1^2 e^{\sin z}\cos z\,dz.$ $= \int_1^2 \frac{d}{dz} e^{\sin z} = e^{\sin 2} - e^{\sin 1}$