Solutions

11/16/11 **MA 525** ______________________

(25) 1. These questions require short answers. Always justify your answer with a reason or an example.

(*a*). If f is analytic at $z = 0$ and $f(0) = 0$, can e^f have a pole at $z = 0$?

No. e^f is bounded near 0, so has a removable singularity there

(*b*). Let f be analytic and non-constant in $\{|z| < 1\}$. Can f (that is, $|f|$) have a minimum at $z = 0$?

Yes $f(z) = z^2$ for $|z| < 1$ for example.

(*c*). If f is analytic, non-constant in $\{|z| < 1\}$, can f (that is, $|f|$) have a maximum at $z = 0$?

No Theorem in class. Ch 4! Maximum principle.

(*d*). If f is analytic in $\{|z| < 1\}$ and $f(1/2) = f(2/3) = f(3/4) = \cdots = 0$, must f be identically zero?

If f is analytic at $z=1$, answer is Yes, but hypothesis doesn't say that so answer is No.

(*e*). If $f(z) = \sum_n a_n z^n$ converges at $z = 1$, must it converge at $z = i$?

No if R = rad of convergence = 1, then depends on each individual point.

(*f*). If f is analytic in $0 < |z - z_0| < 1$ and $\lim f(z)$ exists as $z \to z_0$ in the direction parallel to the x-axis, can f have an essential singularity at z_0?

Yes $z \sin \frac{1}{z}$ is an example. But knowing there is a limit on part of approach says nothing unless we have pole or remov. sing.

(15) 2. Find $\text{Residue}_{z=0}\, z^{-4} \sin z$.

$$z^{-4} \sin z = \frac{1}{z^4}\left(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots\right)$$

$$\text{So res} = \text{coeff of } \frac{1}{z} = -\frac{1}{3!} = -\frac{1}{6}$$

(20) 3. Find

$$\int_{\{|z-2|=2/3\}} \frac{z}{(z-1)(z-2)^2}\, dz$$

Show all steps.

$z=1$ is only singularity

Answer is

$$2\pi i \left(\frac{d}{dz} \frac{z}{z-1}\right)_{z=1}$$

(computation omitted)

(15) 4. Expand the function $1/(1-z)$ in powers of z such that it converges when $z = 2$. Be sure to indicate precisely in what region the series converge.

Answer has to be in terms of <u>powers of z</u>; not $z-2$, for example. So

$$\frac{1}{1-z} = -\frac{1}{z} \cdot \frac{1}{1-1/z} = -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

$$= -\sum_{-\infty}^{-1} z^k$$

Converges in largest annulus containing $z=2$, centered at 0, in which $\frac{1}{1-z}$ is analytic. So converges for

$$1 < |z| < \infty$$

(25) 5. In the book's problem set is the challenge: to show that $f(x)$

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x}{(x^2+4x+13)^2}\, dx = -\frac{\pi}{27}\, dx.$$

(*a*) Explain why this is a normal convergent improper integral, rather than just a 'principal value'.

Function continuous on $\mathbb{R}$ and
$|f(x)| \leq \frac{C}{|x|^3}$ at ∞ ($\frac{C}{|x|^k}$ for any $k > \cancel{1}$ is enough)

(*b*) Expain why we can compute this integral by finding only

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x}{(x^2+4x+13)^2}\, dx$$

Because def of improper integral is
$\lim_{M,N\to\infty} \int_{-M}^{N} f(x)\,dx$. But since from (a) we know this limit exists, we can let $M, N \to \infty$ in any convenient way. Here we take $M = N = R$.

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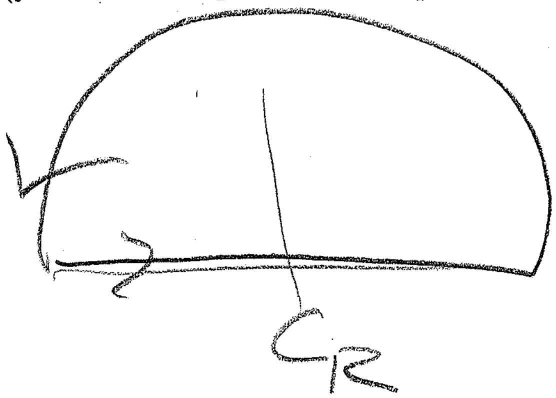
(*c*) Indicate what contours you will use to compute this integral, show what function is being integrated, and compute the residue which will appear when finding the answer (you are not expected to compute the integral).



Poles (double) at

$z = -2 \pm 3i$

Only $-2+3i$ is inside,

So answer is

$$2\pi i \left(\frac{d}{dz} \left(\frac{z}{(z-(-2-3i))} \right) \Big|_{z=3i} \right)$$