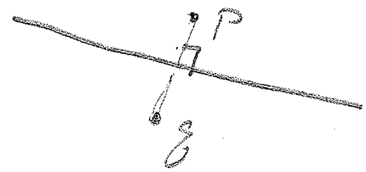
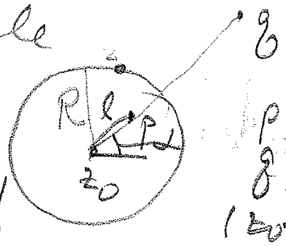
Math 525 — Linear fractional transformations.

<u>Symmetric points</u> C a circle; p, q points in $\mathbb{C}$.

(a) C a line:



(b) C a circle



Let $z = z_0 + Re^{i\theta} \in C$. Then

$$\left|\frac{z-p}{z-q}\right| = \left|\frac{Re^{i\theta}-\ell e^{i\alpha}}{Re^{i\theta}-(R^2/\ell)e^{i\alpha}}\right| = \frac{\ell}{R}\left|\frac{Re^{i\theta}-\ell e^{i\alpha}}{\ell e^{i\theta}-Re^{i\alpha}}\right| = \frac{\ell}{R}$$

$p - z_0 = \ell e^{i\alpha}$

$q - z_0 = \frac{R^2}{\ell} e^{i\alpha}$

($z_0 \leftrightarrow \infty$)

Let p, q be symmetric with respect to C and

(*) $w = \frac{az+b}{cz+d}$, $z = \frac{dw-b}{-cw+a}$

a linear fractional transformation. Then $w(p)$ and $w(q)$ are symmetric with respect to $w(C)$.

Let C be given by

$$\left|\frac{z-p}{z-q}\right| = k \quad \text{(with, say, } k \neq 1\text{)}.$$

Then if we use the formula (*) (2nd one),

$$\left|\frac{dw-b-p(-cw+a)}{dw-b-q(-cw+a)}\right| = k,$$

$$\left|\frac{w(cp+d)-(ap+b)}{w(cq+d)-(aq+b)}\right| = k,$$

$$\left|\frac{w - \frac{ap+b}{cp+d}}{w - \frac{aq+b}{cq+d}}\right| = k\left|\frac{cq+d}{cp+d}\right|,$$

a circle.