MA525 ON CAUCHY'S THEOREM AND GREEN'S THEOREM

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(EDITED BY JOSIAH YODER)

1. Introduction

No doubt the most important result in this course is Cauchy's theorem. Every critical theorem in the course takes advantage of it, and it is even used to show that all analytic functions must have derivatives of all orders.[1]

There are many ways to formulate it, but the most simple, direct and useful is this: *Let f be analytic inside the simple closed curve γ. Then*

$$\int_\gamma f(z)dz = 0. \tag{1.1}$$

The most natural way to prove this is by using Green's theorem. We state the conclusion of Green's theorem now, leaving a discussion of the hypotheses and proof for later. The formula reads: *D is a region bounded by a system of curves γ (oriented in the 'positive' direction with respect to D) and P and Q are functions defined on $D \cup \gamma$. Then*

$$\int_\gamma Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy. \tag{1.2}$$

Green's theorem leads to a trivial proof of Cauchy's theorem. Although this is only a formal[2] proof, since we have not discussed the conditions necessary to apply Green's theorem, I think it is impressive how 'simple' and natural the proof becomes:

$$f = u + iv \quad dz = dx + idy \tag{1.3}$$

and then[3]

$$\int_\gamma f(z)dz = \int_\gamma (u+iv)(dx+idy) = \int_\gamma u\,dx - v\,dy + i\int_\gamma u\,dy + v\,dx. \tag{1.4}$$

If we apply Green's theorem to each of these line integrals,

$$= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy, \tag{1.5}$$

and use the Cauchy-Riemann equations[4]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{1.6}$$

we see that the integrand in each double integral is (identically) zero. In this sense, Cauchy's theorem is an immediate consequence of Green's theorem.

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[1] Recall that a complex function is *analytic* in a region if its first-order derivative exists.

[2] "formal" in the sense that it shows the general outline, but not the details. Not "formal" in the sense of "dressed up for the ball" or "rigorous."

[3] Recall that we define a complex integral along a contour as $\int_\gamma f(z)dz = \int_{t_0}^{t_1} f(z(t))(dz/dt)dt$ where $z(t)$ is a parameterization of the path γ. Thus $\int_\gamma (u+iv)(dx+idy) = \int_{t_0}^{t_1}(u+iv)((dx/dt)+i(dy/dt))\,dt$ is a standard definite integral and nothing to be afraid of! As an example, consider the integral on the left side of Green's Theorem: $\int_\gamma Pdx + Qdy = \int_{t_0}^{t_1} P(dx/dt) + Q(dy/dt)\,dt$. From this it is clear that we can split it into two integrals: $\int_\gamma Pdx + Qdy = \int_\gamma Pdx + \int_\gamma Qdy$.

[4] We will review a proof of the Cauchy-Riemann equations as part of Thm. 3 on page 4.

In fact, Green's theorem is itself a fundamental result in mathematics — the fundamental theorem of calculus in higher dimensions. Proofs of Green's theorem are in all the calculus books, where it is always assumed that P and Q have *continuous partial derivatives.* Using these restricts our simple proof above to functions with continuous partial derivatives as well. Unless Cauchy's Theorem applies for all analytic functions, it cannot be used as the basis for the many important theorems derived from it in this course.

This note makes a case for the simple, elegant proof above by demonstrating that Green's theorem applies to all analytic functions, not just functions with continuous partial derivatives.

The heart of this proof is a variation on E. Goursat's 'elementary' proof of Cauchy's theorem. The other observations are not original either, but I am collecting them together for your convenience.

Editor's Note: Though this document is only 8 pages or so, it will probably take two sittings to read through it. I'll give you a warning when I think it's time for break!

2. What is wrong?

There are two possible objections to the proof I just presented. One, which we consider only briefly here, is that we have not carefully described what kind of curves we are allowing, or what we mean by the 'positive' direction of circuiting γ. We shall allow only rectangular regions where the positive direction may be clearly defined. Extending the proof to other regions is a problem of point-set topology or geometric measure theory, and this note offers no insight on these issues.[5]

The second, and principle, objection is that we have not stated the hypotheses on P and Q needed to apply Green's theorem. Supplying these hypotheses, and a proof that Green's theorem still holds, is the purpose of this note.

As mentioned above, the proofs of Green's theorem in the calculus books assume that the partial derivatives are continuous. When applied to our analytic function $f(z)$, this means that we are assuming that the partial derivatives $u_x(= \partial u/\partial x)$, u_y, v_x and v_y are continuous. The partial derivatives of an analytic function are continuous, but this is something that is most often proved using Cauchy's Theorem. To avoid circular reasoning, a proof of Cauchy's Theorem must not make this assumption.

The purpose of this note is to show that we do not need to assume P and Q have continuous partials; indeed, Green's theorem holds when P and Q satisfy conditions which fit *exactly* with what it means for $f = P + iQ$ to be analytic.

3. Outline

Our goal is to prove a general form of Cauchy's Theorem:

Theorem 1. *Let f be analytic inside a rectangle R and continuous on its boundary. Then Cauchy's theorem (1.1) holds.*

We will do this using the techniques of Section 1 with a formulation of Green's Theorem which does not depend on continuous partial derivatives:

Theorem 2. *Let P and Q be differentiable inside and on a rectangle R with boundary γ and suppose that*

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \tag{3.1}$$

Then Green's Theorem (1.2) holds.

[5] Perhaps this could be the subject of a second note? Or an appendix to the note?

The proof for this theorem will be presented in Section 8. Note that $\partial Q/\partial x - \partial P/\partial y$ could be identically zero without the component terms $\partial Q/\partial x$ and $\partial P/\partial y$ being continuous.

To be able to use Theorem 2 to derive Theorem 1, we will check two things:

(1) That u and v are differentiable at each point z at which $f'(z)$ exists. To aid in this endeavor, we will review what it means to be differentiable.
(2) That $\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$. This follows from the Cauchy-Riemann equations, which we will also review.

Verifying these points will also prepare us for the final step, proving Green's Theorem (Thm 2). For this, we will use the same argument that E. Goursat introduced to give his famous 'elementary' proof of Cauchy's theorem, which appeared in Volume 1 of the *Transactions of the American Mathematical Society.*

4. How do we prove Thm 1?

Once we have proved Green's theorem, as stated in Thm. 2, we can apply the proof given in the first section of this paper. But how do we know the combination $\partial Q/\partial x - \partial P/\partial y$ is identically zero?

This is because the Cauchy-Riemann equations hold. To be specific, we know that if f is analytic, then $\partial u/\partial x - \partial v/\partial y = 0$. Similarly, $-\partial v/\partial x - \partial u/\partial y = 0$. Thus one of the conditions required for Thm. 2 is met perfectly by any analytic function. The remaining condition is that P and Q be differentiable.

If you are feeling a little uncertain about the Cauchy-Riemann equations, don't despair! We will actually prove these on the side in Section 6.

5. Differentiability

Our version of Green's Theorem requires that the P and Q be differentiable. Before we can show that the real and imaginary parts of any analytic function are differentiable, a little review of differentiability is in order. Although this is standard material in third-semester calculus, the review is likely helpful. In dimensions greater than one, being differentiable is a much stronger property than having partial derivatives, since it is a condition on the entire region surrounding a point p, not just the paths of constant x or y through p. To see how this is, let us consider the definition of differentiability for a real-valued function u in the domain D.

Definition. The function $u(x, y)$ is *differentiable* at (a, b) if there are constants A and B so that

$$u(x,y) - u(a,b) = A\,(x-a) + B\,(y-b) + S(x,y) \tag{5.1}$$

where the 'remainder' S satisfies

$$\lim_{(x,y)\to(a,b)} \frac{S(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0. \tag{5.2}$$

Intuitively, this means that a function is differentiable if it can be locally approximated by a linear function with a remainder term that vanishes near the point (a, b).

Based on this intuition, we might guess that every differentiable function has partial derivatives. Proving this makes a good exercise, as follows. If we set y identically equal to b, and let $x \to a$, then $\sqrt{(x-a)^2+(y-b)^2} = \sqrt{(x-a)^2} = |x-a|$, so (5.2) tells us that

$$\lim_{x\to a} \frac{u(x,b) - u(a,b) - A\cdot(x-a)}{|x-a|} = 0. \tag{5.3}$$

It is quite amazing how useful this formulation is. Since the limit on the right side is zero, we can multiply the left side by any factor of absolute value one without changing the equation. Multiplying by $|x-a|/(x-a)$, we have at once

$$\lim_{x\to a}\frac{u(x,b)-u(a,b)-A\cdot(x-a)}{(x-a)}=0: \tag{5.4}$$

$$u_x(a,b)=A; \tag{5.5}$$

similarly, we see that $u_y(a,b)=B$. So it is not difficult to show that a differentiable function has partial derivatives.

However, as we mentioned at the beginning, being differentiable is a much stronger property than having partial derivatives, since it is a condition independent of how $(x,y)\to(a,b)$. For example, the function $u(x,y)=xy/(x^2+y^2)$ has $u_x(0,0)=u_y(0,0)=0$, but it is not differentiable, or even continuous, at (0,0) since $u(x,x)=1/2$ on the 45-degree line through the origin, but is zero on both the axes.

Before leaving this section, let's note that differentiability is defined similarly for functions of complex variables. The function $u(z)=u(x,y)$ is differentiable if there are constants A and B such that

$$u(z)-u(z_0)=A\cdot(x-a)+B\cdot(y-b)+S(z) \tag{5.6}$$

where the remainder S satisfies

$$\lim_{z\to z_0}\frac{S(z)}{|z-z_0|}=0. \tag{5.7}$$

We can imagine this formula intuitively in the same way as the first formula, since $S(z)$ can be imagined as a surface. Because z is a complex number, its real and imaginary parts take the place of x and y in our previous discussion.

6. First Blood

Let us prove a little theorem. The proof is not hard at all, and if you go through it, you will see that you are just rearranging equalities everywhere. And for this proof, the moral of the story is critical to our argument; it says that for u and v to be differentiable is as natural as $f=u+iv$ having a derivative (that is, as natural as f being analytic), meeting the condition that P and Q are differentiable in Thm. 2.

Theorem 3. *Let $f=u+iv$ be defined in some neighborhood of $z_0=a+ib$. Then f' exists at z_0 if and only if, at z_0, we have both that u and v are differentiable, and that the partials of u and v satisfy $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ (the Cauchy-Riemann equations).*

Proof. First let's assume that the derivative exists and is $f'(z_0)=A+iB$. We show that u and v are differentiable and satisfy Cauchy-Riemann. Let's write the difference between the function and its local linear approximation $f(z)-f(z_0)-(A+iB)\cdot(z-z_0)$ in terms of u, v, x, and y. We also separate real and imaginary parts to yield:

As in Sec. 5, we may divide by $z-z_0$ or $|z-z_0|$ as we wish. If we divide by $z-z_0$, the left side tends to 0 since $A+iB=f'(z_0)$. Thus the right-hand side must go to zero as well. Now if we multiply the right-hand side by $(z-z_0)/|z-z_0|$, it must still go to zero, in both the real and imaginary parts. Since the real part has 0 as a limit, u must be differentiable at z_0. Similarly for the imaginary part, v must be differentiable at z_0 as well. Moreover, applying the same technique we used in Section 5, we can show that the numbers A and B are the partial derivatives of both u and v, by considering either the real or the imaginary part. Thus we can show that

$$A=(\partial u/\partial x)|_{z=z_0}=(\partial v/\partial y)|_{z=z_0}, \tag{6.1}$$

$$B=(\partial v/\partial x)|_{z=z_0}=-(\partial u/\partial y)|_{z=z_0}: \tag{6.2}$$

the Cauchy-Riemann equations hold if $f'(z)$ exists at z_0.

The proof in the other direction is just as easy. Let's assume that u and v are differentiable at z_0 and the partials of u and v at that point satisfy the Cauchy-Riemann equations. Then we can denote them as[6]

$$u(z) - u(z_0) = A \cdot (x-a) + B \cdot (y-b) + S \tag{6.3}$$

$$v(z) - v(z_0) = -B \cdot (x-a) + A \cdot (y-b) + T \tag{6.4}$$

where S and T are the remainder terms. We substitute these into $f = u + iv$:

$$\begin{aligned} f(z) - f(z_0) &= (u(z) + iv(z)) - (u(z_0) + iv(z_0)) \\ &= A \cdot (x-a) + B \cdot (y-b) + S + i[-B \cdot (x-a) + A \cdot (y-b) + T] \\ &= A \cdot (x-a) + B \cdot (y-b) + i[-B \cdot (x-a) + A \cdot (y-b)] + S + iT \\ &= (A + iB) \cdot (z - z_0) + S + iT. \end{aligned}$$

Thus, on dividing by $z - z_0$ or $|z - z_0|$ as appropriate, taking the limit as $z \to z_0$, and recalling (5.7) we have that $f'(z_0) = A + iB$, and thus exists. □

7. Taking Stock

We have shown that if f is analytic, then u, v are differentiable and $u_x = v_y$ and $u_y = -v_x$, such that $\partial Q/\partial x - \partial P/\partial y = 0$ in both applications of Green's theorem. Therefore, we can apply the analytic-conditions version of Green's theorem (Thm. 2) to prove the analytic-conditions version of Cauchy's theorem (Thm. 1). All that remains is the proof for Thm. 2.

8. Final Advance

Editor's Note: At this point, you may want to set down this paper for a while, and perhaps come back to it tomorrow. We have covered a lot of ground, and even I (Josiah) am usually pretty tired by the time I read to this point. But the most beautiful part of the proof is in this section, so I hope you come back!)

For linear functions, that is, functions of the form $f(x, y) = Ax + By + c$, we already have a proof of Green's theorem (1.2), because linear functions have continuous partial derivatives, so the standard proofs of Green's theorem apply. (Alternatively, we can demonstrate Green's theorem directly for linear equations. This *will be* done in Appendix A*, but the appendix isn't finished yet.*)

To prove Green's theorem (Thm. 2) for general functions, we use a technique similar to Goursat's. Let us suppose that Green's theorem is applied to a rectangular region R_0 with boundary γ_0; then we wish to show that

$$\int_{\gamma_0} Pdx + Qdy - \iint_{R_0} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dxdy = 0 \tag{8.1}$$

Since $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$, the double integral is zero, and we need only show that

$$\int_{\gamma_0} Pdx + Qdy = 0 \tag{8.2}$$

Suppose this is false, then there must be a Δ_0 such that

$$\left| \int_{\gamma_0} Pdx + Qdy \right| > \Delta_0 \tag{8.3}$$

[6]Again, can use the result of Section 5 to show that the constants A and B correspond to the partial derivatives, and apply the Cauchy-Riemann equations to show that the constants for v must be related to the constants for u as shown here.

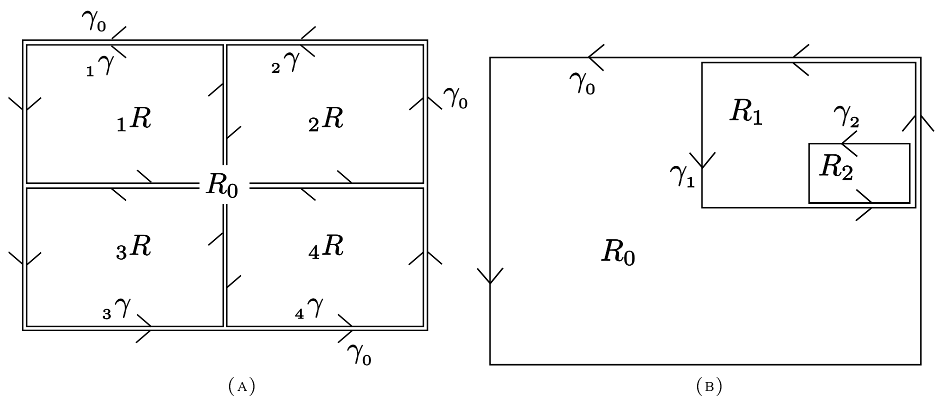

FIGURE 8.1. Dividing the region into quadrants. (a) At each level, the rectangular region is divided into quadrants. If the total region has difference Δ_0, at least one of these quadrants must have a difference $\Delta_1 \geq \Delta_0/4$. (b) We continue this, finding successively smaller regions such that $\Delta_n \geq \Delta_0/4^n$. Here, $R_1 = {}_2R_0$ and $R_2 = {}_4R_1$. Note that there are no margins between the rectangles, these are included in the figure to allow the sub-regions to be distinguished more easily.

We will prove that Δ_0 does not exist by contradiction. Here is Goursat's idea. Suppose we divide R_0 into quadrants ${}_1R$, ${}_2R$, ${}_3R$, and ${}_4R$ as illustrated in Figure 8.1a, and look at the integrals[7]

$$\left|\int_{{}_i\gamma} Pdx + Qdy\right|,\ i = 1,2,3,4 \tag{8.4}$$

We can't have all four of these differences less than $\Delta_0/4$, for if they were, we could add them up and the sum would be less than Δ_0. That means that there must be one smaller rectangle, each side of which is half that of R_0, for which the difference in the two terms is at least $\Delta_0/4$. We shall call this rectangle R_1, its border γ_1, and the difference in the two terms $\Delta_1 \geq \Delta_0/4$.

Now you might be wondering why we can add up the line integrals in (8.4) just like the area integrals. This is a good question. The sum of the four quadrant line integrals add to form the line integral around the main rectangle because, along the interior boundaries, the integrals cancel out. For more details on this, see Appendix B, *which isn't written yet.*

Now we repeat this argument with R_1 and divide it into four smaller rectangles (Fig. 8.1b); for one of them, we have that the difference $\Delta_2 \geq \Delta_1/4 \geq \Delta_0/4^2$. We call this R_2, bounded by γ_2. Continuing in this fashion, for each positive integer n we find a rectangle R_n inside $R_{n-1}$with boundary γ_n such that

$$\left|\int_{\gamma_n} Pdx + Qdy\right| = \Delta_n \geq \Delta_0/4^n \tag{8.5}$$

We shall show that for large enough n, no such region exists.

Suppose that such a region could be found for arbitrarily-large n. Then there must be some point z_0 which is in every one of the nested rectangular regions, $z_0 \in R_i$, $i = 0, 1, 2, \ldots$. Of course, we have stated in the requirements for the proof that P and Q must

[7] Note that we use pre-scripts like ${}_1R$ to represent the four quadrants, and post-scripts like R_1 to represent levels of nested quadrants which we will introduce shortly.

be differentiable inside R_0 and on γ_0, so they must be differentiable in the neighborhood of z_0, and we have

$$P(z) = P(z_0) + A \cdot (x - a) + B \cdot (y - b) + S(z)$$
$$Q(z) = Q(z_0) + C \cdot (x - a) + D \cdot (y - b) + T(z) \tag{8.6}$$

where A, B, C, and D are the partial derivatives at $z_0 = a + ib$ and S and T satisfy

$$\lim_{z \to z_0} \frac{S(z)}{|z - z_0|} = 0, \qquad \lim_{z \to z_0} \frac{T(z)}{|z - z_0|} = 0 \tag{8.7}$$

We consider these expansions on each small rectangle R_n where n is large, to produce a different bound for Δ_n than the one above (8.5). At the beginning of this section, we observed that (1.2) is true when P and Q are linear, and so on consulting (8.6), we see that we need only consider the case that $P(z) = S(z)$, $Q(z) = T(z)$. We then look at the line and double integrals in (8.5) separately.

The length of γ_n is $d_n := d_0 2^{-n}$, where d_0 is the length of γ_0. And (8.7) gives that

$$S(z) \le \epsilon |z - z_0| < \epsilon d_0 \tag{8.8}$$

where ϵ may be taken as small as we wish when n is large enough. (A large n guarantees that all points $z \in R_n$ are close enough to z_0). The same bound applies for $T(z)$. From these bounds, we can establish a bound for (8.5)

$$\left| \int_{\gamma_n} P(z)dx + Q(z)dy \right| := \left| \int_{\gamma_n} S(z)dx + T(z)dy \right| \le 2(\epsilon d_n) \cdot d_n = 2\epsilon d_0^2 4^{-n} \tag{8.9}$$

using the upper bound $(\max_{\gamma_n} |S(z)| + |T(z)|) \cdot |\gamma_n|$.

Thus, for any $\epsilon > 0$, an n exists which is so large that

$$\left| \iint_{\gamma_n} P(z)dx + Q(z)dy - \iint_{R_n} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \right| = \Delta_n \le \epsilon 2 d_0^2 4^{-n} = eC_0 4^{-n} \tag{8.10}$$

where $C_0 = 2d_0^2$ is fixed with respect to n and ϵ. This contradicts (8.5), which stated that $\Delta_n \ge \Delta_0 4^{-n}$ for all n.[8]

This contradiction proves that our version of Cauchy's theorem holds precisely under the hypothesis that $f(z)$ has a derivative at each point of our rectangle R and its boundary.

Appendix A: Green's Theorem For Linear Functions

Editor's Note: This section is unfinished, and probably contains a few mistakes. Although the linear integrals are easy, they are difficult to describe with clear notation. If you would find this proof helpful in understanding the whole document, please email yoder2@purdue.edu, and I will attempt a draft.

In our final advance, we will need to know that Green's theorem holds when P and Q are linear functions, that is

$$P(x, y) = A + Bx + Cy \tag{8.11}$$

$$Q(x, y) = D + Ex + Fy \tag{8.12}$$

where A, B, C, D, E, and F are constants. In this case, we can simply use one of the common proof of Green's theorem which only work when P and Q are continuous. But since it is not hard to provide a special proof just for linear functions, we will do that here.

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[8]As an example of the contradiction, suppose that $C* = 0.5$, and that we hypothesize that $\Delta_0 = 0.2$. We can show this $\Delta_0$is impossible by selecting (for example) $\epsilon = \Delta_0 / C^* / 2 = 0.2$ such that $\Delta_n \le (0.1)(4^{-n})$ for sufficiently large n and $\Delta \ge (0.2)(4^{-n})$ for all n, a contradiction. This same procedure could be used to prove that any $\Delta_0 > 0$ will not work. Thus Δ_0 must be 0.

Here, we will consider the case where the γ is a rectangle aligned with the axes with vertices at $(0,0)$ and (α,β) as shown in Figure XXXX. In this case, it is straightforward to check that[9]

$$\int_\gamma A + Bx\,dx = 0 \tag{8.13}$$

$$\int_\gamma D + Ey\,dy = 0 \tag{8.14}$$

so that

$$\int_\gamma P(x,y)dx - \int_\gamma Q(x,y)dy = \int_\gamma Cy\,dx - \int_\gamma Fx\,dy \tag{8.15}$$

We evaluate the integrals on the right-hand side. You may wish to do this on a separate piece of paper. To do this, it helps to define the boundaries of the rectangle with coordinates *[Unfinished]*

Appendix B

Editor's Note: This Appendix will go into detail on why we can add contour integrals of the partitions of a region to get the contour integral of the whole. This is an important concept to understand for any student of Complex Analysis. If you would find a review of this subject helpful, please email Josiah Yoder at yoder2@purdue.edu

[9]Actually, we don't even need to assume it's on a rectangle, but since our final proof is for rectangular regions, we will leave it at that. See footnote 3 on page 1 for a hint on how to get started with more general forms of γ.