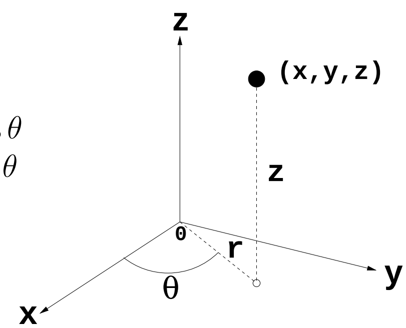
Study Guide # 3

You also need Study Guides # 1 and # 2 for the Final Exam

1. Cylindrical Coordinates (r, θ, z):

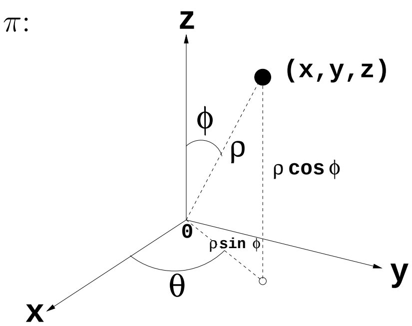
From CC to RC : $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$



Going from RC to CC use $x^2 + y^2 = r^2$ and $\tan\theta = \dfrac{y}{x}$ (make sure θ is in correct quadrant).

2. Spherical Coordinates (ρ, θ, ϕ), where $0 \le \phi \le \pi$:

From SC to RC : $\begin{cases} x = (\rho\sin\phi)\cos\theta \\ y = (\rho\sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases}$



Going from RC to SC use $x^2 + y^2 + z^2 = \rho^2$, $\tan\theta = \dfrac{y}{x}$ and $\cos\phi = \dfrac{z}{\rho}$.

3. Triple integrals in Cylindrical Coordinates: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$, $\boldsymbol{dV = r\,dz\,dr\,d\theta}$

$$\iiint_E f(x, y, z)\ dV = \iiint_E f(r\cos\theta, r\sin\theta, z)\, \underset{\uparrow}{r}\, dz\, dr\, d\theta$$

4. Triple integrals in Spherical Coordinates: $\begin{cases} x = (\rho\sin\phi)\cos\theta \\ y = (\rho\sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases}$, $\boldsymbol{dV = \rho^2 \sin\phi \ \ d\rho\, d\phi\, d\theta}$

$$\iiint_E f(x, y, z)\ dV = \iiint_E f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\, \underset{\uparrow}{\rho^2\sin\phi}\, d\rho\, d\phi\, d\theta$$

5. Vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$: $\vec{\mathbf{F}}(x, y) = \langle P(x, y),\ Q(x, y)\rangle$ and $\vec{\mathbf{F}}(x, y, z) = \langle P(x, y),\ Q(x, y),\ R(x, y)\rangle$;
$\vec{\mathbf{F}}$ is a conservative vector field if $\vec{\mathbf{F}} = \nabla f$, for some real-valued function f.

6. Line integral of a function $f(x,y)$ along C, parameterized by $x = x(t), \quad y = y(t)$ and $a \leq t \leq b$, is

$$\int_C f(x,y)\ ds = \int_a^b f(x(t),\, y(t))\ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt\,.$$

(independent of orientation of C, other properties and applications of line integrals of f)

<u>Remarks</u>:

(a) $\int_C f(x,y)\,ds$ is sometimes called the *"line integral of f with respect to arc length"*

(b) $\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$

(c) $\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$

7. Line integral of vector field $\vec{\mathbf{F}}(x,y)$ along C, parameterized by $\vec{\mathbf{r}}(t)$ and $a \leq t \leq b$, is given by

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\,dt\,.$$

(depends on orientation of C, other properties and applications of line integrals of f)

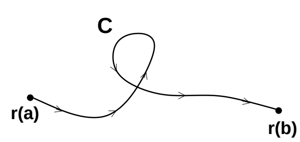
8. Connection between line integral of vector fields and line integral of functions:

$$\int_C \vec{\mathbf{F}}\ \cdot\ d\vec{\mathbf{r}} = \int_C (\vec{\mathbf{F}} \cdot \vec{\mathbf{T}})\,ds$$

where $\vec{\mathbf{T}}$ is the unit tangent vector to the curve C.

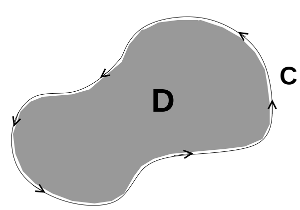
9. If $\vec{\mathbf{F}}(x,y) = P(x,y)\,\vec{\mathbf{i}} + Q(x,y)\,\vec{\mathbf{j}}$, then $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C P(x,y)\,dx + Q(x,y)\,dy$; Work $= \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

10. <u>Fundamental Theorem of Calculus for Line Integrals</u>: $\int_C \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$:



11. A vector field $\vec{\mathbf{F}}(x,y) = P(x,y)\,\vec{\mathbf{i}} + Q(x,y)\,\vec{\mathbf{j}}$ is *conservative* (i.e. $\vec{\mathbf{F}} = \nabla f$) if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$; how to determine a potential function f if $\vec{\mathbf{F}}(\vec{\mathbf{x}}) = \nabla f(\vec{\mathbf{x}})$.

12. <u>Green's Theorem</u>: $\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA$ ($C =$ boundary of D):



13. Del Operator: $\dfrac{\partial}{\partial x}\,\vec{\mathbf{i}} + \dfrac{\partial}{\partial y}\,\vec{\mathbf{j}} + \dfrac{\partial}{\partial z}\,\vec{\mathbf{k}}$; if $\vec{\mathbf{F}}(x,y,z) = P(x,y,z)\,\vec{\mathbf{i}} + Q(x,y,z)\,\vec{\mathbf{j}} + R(x,y,z)\vec{\mathbf{k}}$, then

$$\text{curl } \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad \text{and} \quad \text{div } \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

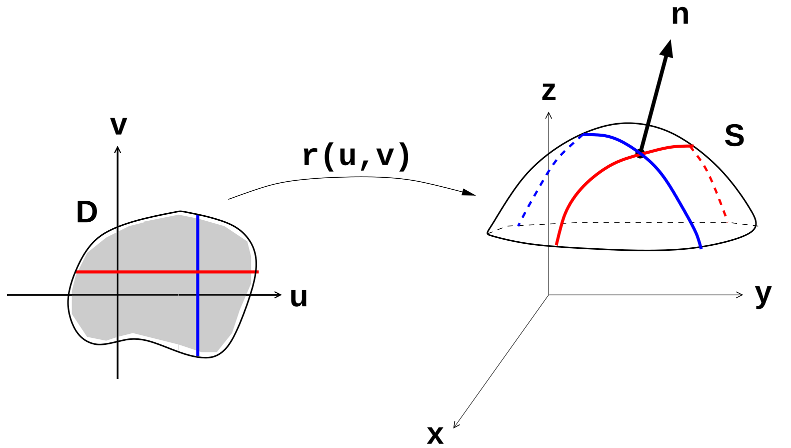
Properties of curl and divergence:

(i) If curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is a conservative vector field (i.e., $\vec{\mathbf{F}}(\vec{\mathbf{x}}) = \nabla f(\vec{\mathbf{x}})$).

(ii) If curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is *irrotational*; if div $\vec{\mathbf{F}} = 0$, then $\vec{\mathbf{F}}$ is *incompressible*.

(iii) *Laplace's Equation*: $\nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0$.

14. Parametric surface S: $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, where $(u,v) \in D$:



Normal vector to surface S : $\vec{\mathbf{n}} = \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$; tangent planes and normal lines to parametric surfaces.

15. Surface area of a surface S:

(i) $A(S) = \displaystyle\iint_D |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, dA$

(ii) If S is the graph of $z = f(x,y)$ above D, then $A(S) = \displaystyle\iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$;

Remark: $dS = |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, dA =$ differential of surface area; while $d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA$

16. The surface integral of $f(x,y,z)$ over the surface S:

(i) $\displaystyle\iint_S f(x,y,z)\, dS = \iint_D f(\vec{\mathbf{r}}(u,v))\, |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, dA.$

(ii) If S is the graph of $z = h(x,y)$ above D, then

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,h(x,y)) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA.$$

17. The surface integral of $\vec{\mathbf{F}}$ over the surface S (recall, $d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\ dA$):

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA.$$

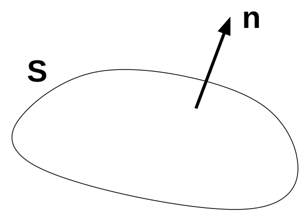
$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}})\ dS = \iint_D \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA.$$

(i) Connection between surface integral of a vector field and a function:

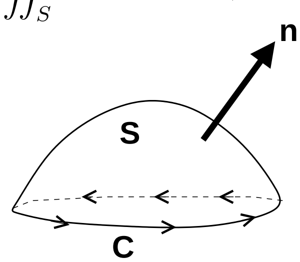
$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}})\ dS.$$

(The above gives another way to compute $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$)

(ii) $\displaystyle\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}})\ dS =$ *flux* of $\vec{\mathbf{F}}$ across the surface S.



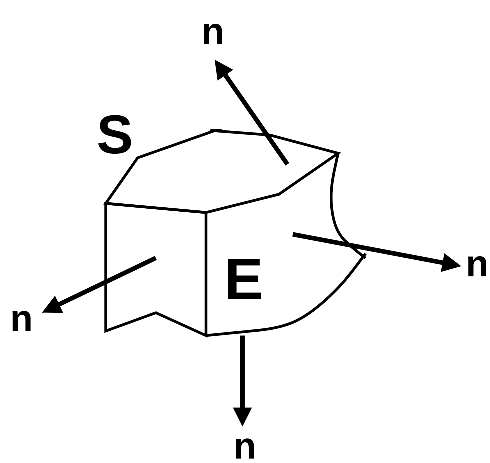
18. STOKES' THEOREM: $\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S \operatorname{curl} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ (recall, $\operatorname{curl} \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}}$).



$\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} =$ *circulation* of $\vec{\mathbf{F}}$ around C.

19. THE DIVERGENCE THEOREM/GAUSS' THEOREM: $\displaystyle\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iiint_E \operatorname{div} \vec{\mathbf{F}}\ dV$

(recall, $\operatorname{div} \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}}$).



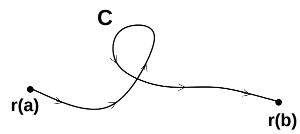
20. Summary of Line Integrals and Surface Integrals:

LINE INTEGRALS	SURFACE INTEGRALS
$C : \vec{\mathbf{r}}(t)$, where $a \le t \le b$	$S : \vec{\mathbf{r}}(u,v)$, where $(u,v) \in D$
$ds = \lvert\vec{\mathbf{r}}\,'(t)\rvert\, dt =$ differential of arc length	$dS = \lvert\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\rvert\, dA =$ differential of surface area
$\int_C ds =$ length of C	$\iint_S dS =$ surface area of S
$\int_C f(x,y,z)\, ds = \int_a^b f(\vec{\mathbf{r}}(t))\, \lvert\vec{\mathbf{r}}\,'(t)\rvert\, dt$ (independent of orientation of C)	$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{\mathbf{r}}(u,v))\, \lvert\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\rvert\, dA$ (independent of normal vector $\vec{\mathbf{n}}$)
$d\vec{\mathbf{r}} = \vec{\mathbf{r}}\,'(t)\, dt$	$d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\ dA$
$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t)\, dt$ (depends on orientation of C)	$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v)) \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\ dA$ (depends on normal vector $\vec{\mathbf{n}}$)
$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \left(\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\right)\ ds$ The *circulation* of $\vec{\mathbf{F}}$ around C	$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S \left(\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\right)\ dS$ The *flux* of $\vec{\mathbf{F}}$ across S in direction $\vec{\mathbf{n}}$

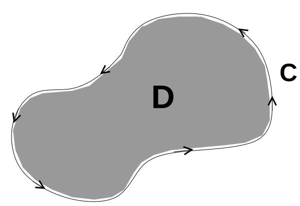
21. Integration Theorems:

<u>Fundamental Theorem of Calculus</u>: $\int_a^b F'(x)\,dx = F(b) - F(a)$

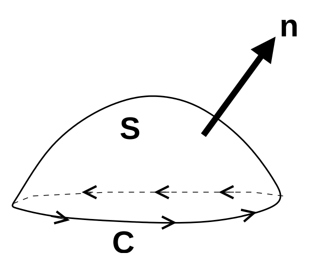
<u>Fundamental Theorem of Calculus For Line Integrals</u>: $\int_a^b \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$



<u>Green's Theorem</u>: $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\, dA = \int_C P(x,y)\,dx + Q(x,y)\,dy$



<u>Stokes' Theorem</u>: $\iint_S \operatorname{curl} \vec{\mathbf{F}} \cdot\, d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot\, d\vec{\mathbf{r}}$



<u>Divergence Theorem</u>: $\iiint_E \operatorname{div} \vec{\mathbf{F}}\;\, dV = \iint_S \vec{\mathbf{F}} \cdot\, d\vec{\mathbf{S}}$