16. The following initial-value problem arises in the analysis of a cable suspended between two fixed points

$$y'' = \frac{1}{a}\sqrt{1+(y')^2}, \quad y(0) = a, \quad y'(0) = 0,$$

where a is a nonzero constant. Solve this initial-value problem for $y(x)$. The corresponding solution curve is called a **catenary**.

17. Consider the general second-order linear differential equation with dependent variable missing:

$$y'' + p(x)y' = q(x).$$

Replace this differential equation with an equivalent pair of first-order equations and express the solution in terms of integrals.

18. Consider the general third-order differential equation of the form

$$y''' = F(x, y''). \tag{1.11.27}$$

(a) Show that Equation (1.11.27) can be replaced by the equivalent first-order system

$$\frac{du_1}{dx} = u_2, \quad \frac{du_2}{dx} = u_3, \quad \frac{du_3}{dx} = F(x, u_3),$$

where the variables u_1, u_2, u_3 are defined by

$$u_1 = y, \quad u_2 = y', \quad u_3 = y''.$$

(b) Solve $y''' = x^{-1}(y'' - 1)$.

19. A simple pendulum consists of a particle of mass m supported by a piece of string of length L. Assuming that the pendulum is displaced through an angle θ_0 radians from the vertical and then released from rest, the resulting motion is described by the initial-value problem

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0, \quad \theta(0) = \theta_0, \quad \frac{d\theta}{dt}(0) = 0. \tag{1.11.28}$$

(a) For small oscillations, $\theta << 1$, we can use the approximation $\sin\theta \approx \theta$ in Equation (1.11.28) to obtain the linear equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0, \quad \theta(0) = \theta_0, \quad \frac{d\theta}{dt}(0) = 0.$$

Solve this initial-value problem for θ as a function of t. Is the predicted motion reasonable?

(b) Obtain the following first integral of (1.11.28):

$$\frac{d\theta}{dt} = \pm\sqrt{\frac{2g}{L}(\cos\theta - \cos\theta_0)}. \tag{1.11.29}$$

(c) Show from Equation (1.11.29) that the time T (equal to one-fourth of the period of motion) required for θ to go from 0 to θ_0 is given by the *elliptic integral of the first kind*

$$T = \sqrt{\frac{L}{2g}}\int_0^{\theta_0} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\, d\theta. \tag{1.11.30}$$

(d) Show that (1.11.30) can be written as

$$T = \sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2\sin^2 u}}\, du,$$

where $k = \sin(\theta_0/2)$. [**Hint:** First express $\cos\theta$ and $\cos\theta_0$ in terms of $\sin^2(\theta/2)$ and $\sin^2(\theta_0/2)$.]

1.12 Chapter Review

Basic Theory of Differential Equations

This chapter has provided an introduction to the theory of differential equations. A **differential equation** involves one or more derivatives of an unknown function, and the highest-order derivative is the **order** of the differential equation.

For an nth-order differential equation, the **general solution** contains n arbitrary constants, and all solutions can be obtained by assigning appropriate values to the constants. This chapter is concerned mainly with **first-order differential equations**, which may be written in the form

$$\frac{dy}{dx} = f(x, y), \tag{1.12.1}$$

for some given function f. If we impose an **initial condition** specifying the value of a solution $y(x)$ to the differential equation (1.12.1) at a particular point x_0, say $y_0 = y(x_0)$, then we have an **initial-value problem**:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0. \tag{1.12.2}$$

To solve an initial-value problem of the form (1.12.2), the first step is to determine the general solution to the differential equation (1.12.1), and then use the initial condition to determine the specific value of the arbitrary constant appearing in the general solution.

Solution Techniques for First-Order Differential Equations

One of our main goals in this chapter is to find solutions to first-order differential equations of the form (1.12.1). There are various ways in which we can seek these solutions:

1. **Geometrically:** The function $f(x, y)$ gives the slope of the tangent line to the solution curves of the differential equation (1.12.1) at the point (x, y). Thus, by computing $f(x, y)$ for various points (x, y), we can draw small line segments through the point (x, y) with slope $f(x, y)$ to depict how a solution curve would pass through (x, y). The resulting picture of line segments is called the **slope field** of the differential equation, and any solution curves to the differential equation in the xy-plane must be tangent to the slope field at all points.

   For example, the differential equation $dy/dx = -x/y$ determines a slope field consisting of small line segments that encircle the origin. Indeed, the solutions to this differential equation consist of concentric circles centered at the origin.

   One piece of theory is that different solution curves for the same differential equation can never cross (this essentially tells us that an initial-value problem cannot have multiple solutions). Thus, for example, if we find a solution to the differential equation (1.12.1) of the form $y(x) = y_0$, for some constant y_0 (recall that such a solution is called an **equilibrium solution**), then all other solution curves to the differential equation must lie entirely above the line $y = y_0$ or entirely below it.

2. **Numerically:** Suppose we wish to approximate the solution to the initial-value problem (1.12.2) at the point $x = x_1 = x_0 + h$, where h is small. **Euler's method** uses the slope of the solution at (x_0, y_0), which is $f(x_0, y_0)$, to use a tangent line approximation to the solution:

   $$y(x) = y_0 + f(x_0, y_0)(x - x_0).$$

   Therefore, we approximate

   $$y(x_1) = y_0 + f(x_0, y_0)(x_1 - x_0) = y_0 + hf(x_0, y_0).$$

   Now, starting from the point $(x_1, y(x_1))$, we can repeat the process to find approximations to the solutions at other points $x_2, x_3, \dots$. The conclusion is that the approximation to the solution to the initial-value problem (1.12.2) at the points $x_{n+1} = x_0 + nh$ $(n = 0, 1, \dots)$ is

   $$y_{n+1} = y_n + hf(x_n, y_n), \quad n = 0, 1, \dots$$

   In Section 1.10, other modifications to Euler's method are also discussed.

3. **Analytically:** In some situations, we can explicitly obtain an equation for the general solution to the differential equation (1.12.1). These include situations in which the differential equation is separable, first-order linear, first-order homogeneous, Bernoulli, and/or exact. Table 1.12.1 shows the types of differential equations we can solve analytically and summarizes the solution techniques. If a given differential equation cannot be written in one of these forms, then the next step is to try to determine an integrating factor. If that fails, then we might try to find a change of variables that would reduce the differential equation to one of the above types.

Type	Standard Form	Technique
Separable	$p(y)y' = q(x)$	Separate the variables and integrate.
First-order linear	$y' + p(x)y = q(x)$	Rewrite as $\frac{d}{dx}(I \cdot y) = I \cdot q(x)$, where $I = e^{\int p(x)dx}$, and integrate with respect to x.
First-order homogeneous	$y' = f(x, y)$ where $f(tx, ty) = f(x, y)$	Change variables: $y = xV(x)$, and reduce to a separable equation.
Bernoulli	$y' + p(x)y = q(x)y^n$	Divide by y^n and make the change of variables $u = y^{1-n}$. This reduces the differential equation to a linear equation.
Exact	$M\,dx + N\,dy = 0$, with $M_y = N_x$	The solution is $\phi(x, y) = c$, where ϕ is determined by integrating $\phi_x = M$, $\phi_y = N$.

Table 1.12.1: A summary of the basic solution techniques for $y' = f(x, y)$.

Example 1.12.1 Determine which of the above types, if any, the following differential equation falls into:

$$\frac{dy}{dx} = -\frac{(8x^5 + 3y^4)}{4xy^3}.$$

Solution: Since the given differential equation is written in the form $dy/dx = f(x, y)$, we first check whether it is separable or homogeneous. By inspection, we see that it is neither of these. We next check to see whether it is a linear or a Bernoulli equation. We therefore rewrite the equation in the equivalent form

$$\frac{dy}{dx} + \frac{3}{4x}y = -2x^4y^{-3}, \tag{1.12.3}$$

which we recognize as a Bernoulli equation with $n = -3$. We could therefore solve the equation using the appropriate technique. Owing to the y^{-3} term in Equation (1.12.3), it follows that the equation is not a linear equation. Finally, we check for exactness. The natural differential form to try for the given differential equation is

$$(8x^5 + 3y^4)\,dx + 4xy^3\,dy = 0. \tag{1.12.4}$$

In this form, we have

$$M_y = 12y^3, \qquad N_x = 4y^3,$$

so that the equation is not exact. However, we see that

$$(M_y - N_x)/N = 2x^{-1},$$

so that according to Theorem 1.9.11, $I(x) = x^2$ is an integrating factor. Therefore, we could multiply Equation (1.12.4) by x^2 and then solve it as an exact equation. □

Examples of First-Order Differential Equations

There are numerous real-world examples of first-order differential equations. Among the applications discussed in this chapter are Newton's law of cooling, families of orthogonal trajectories, Malthusian and logistic population models, mixing problems, electric circuits, and others.

Additional Problems

1. A racquetball player standing at the back wall of the court hits the ball from a height of 2 feet horizontally toward the front wall at 80 miles per hour. The length of a regulation racquetball court is 40 feet. Does the ball reach the front wall before hitting the ground? Neglect air resistance, and assume the acceleration of gravity is 32 feet/sec^2.

2. A boy 2 meters tall shoots a toy rocket straight up from head level at 10 meters per second. Assume the acceleration of gravity is 9.8 meters/sec^2.

(a) What is the highest point above the ground reached by the rocket?

(b) When does the rocket hit the ground?

In Problems 3–6, find the equation of the orthogonal trajectories to the given family of curves.

3. $y = cx^3$.

4. $y^2 = cx^3$.

5. $y = \ln(cx)$.

6. $x^4 + y^4 = c$.

7. Consider the family of curves

$$x^2 + 3y^2 = 2cy, \tag{1.12.5}$$

(a) Show that the differential equation of this family is

$$\frac{dy}{dx} = \frac{2xy}{x^2 - 3y^2}.$$

(b) Determine the orthogonal trajectories to the family (1.12.5).

In Problems 8–9, sketch the slope field and some representative solution curves for the given differential equation.

8. $y' = \sin x$.

9. $y' = y/x^2$.

10. At time t the velocity, $v(t)$, of an object is governed by the differential equation

$$\frac{dv}{dt} = \frac{1}{2}(25 - v), \quad t > 0.$$

(a) Verify that $v(t) = 25$ is a solution to this differential equation.

(b) Sketch the slope field for $0 \leq v \leq 25$. What happens to $v(t)$ as $t \to \infty$?

11. An object of mass m is released from rest in a medium in which the frictional forces are proportional to the square of the velocity. The initial-value problem that governs the subsequent motion is

$$mv\frac{dv}{dy} = mg - kv^2, \quad v(0) = 0, \tag{1.12.6}$$

where $v(t)$ denotes the velocity of the object at time t, $y(t)$ denotes the distance traveled by the object at time t as measured from the point at which the object was released, and k is a positive constant.

(a) Solve (1.12.6) and show that

$$v^2 = \frac{mg}{k}(1 - e^{-2ky/m}).$$

(b) Make a sketch of v^2 as a function of y.

In Problems 12–37, determine which of the five types of differential equations we have studied the given equation falls into (see Table 1.12.1), and use an appropriate technique to find the general solution.

12. $\dfrac{dy}{dx} = \dfrac{2 \ln x}{xy}$.

13. $xy' - 2y = 2x^2 \ln x$.

14. $\dfrac{dy}{dx} = -\dfrac{2xy}{x^2 + 2y}$.

15. $(y^2 + 3xy + x^2)\, dx - x^2\, dy = 0$.

16. $y' + y(\tan x + y \sin x) = 0$.

17. $\dfrac{dy}{dx} + \dfrac{2e^{2x}}{1 + e^{2x}} y = \dfrac{1}{e^{2x} - 1}$.

18. $y' - x^{-1} y = x^{-1}\sqrt{x^2 - y^2}$.

19. $\dfrac{dy}{dx} = \dfrac{\sin y + y \cos x + 1}{1 - x \cos y - \sin x}$.

20. $\dfrac{dy}{dx} + \dfrac{1}{x} y = \dfrac{25x^2 \ln x}{2y}$.

21. $e^{2x+y}\, dy - e^{x-y}\, dx = 0$.

22. $y' + y \cot x = \sec x$.

23. $\dfrac{dy}{dx} + \dfrac{2e^x}{1 + e^x} y = 2\sqrt{y} e^{-x}$.

24. $y[\ln (y/x) + 1]dx - x dy = 0$.

25. $(1 + 2xe^y)\, dx - (e^y + x)\, dy = 0$.

26. $y' + y \sin x = \sin x$.

27. $(3y^2 + x^2)\, dx - 2xy\, dy = 0$.

28. $2x(\ln x)y' - y = -9x^3 y^3 \ln x$.

29. $(1 + x)y' = y(2 + x)$.

30. $(x^2 - 1)(y' - 1) + 2y = 0$.

31. $x \sec^2(xy)\, dy = -\left[y \sec^2(xy) + 2x\right] dx$.

32. $\dfrac{dy}{dx} - x^2 y = \sqrt{y}$.

33. $\dfrac{dy}{dx} = \dfrac{x^2}{x^2 - y^2} + \dfrac{y}{x}$.

34. $[\ln (xy) + 1]\, dx + \left(\dfrac{x}{y} + 2y\right) dy = 0$.

35. $y' + \dfrac{y}{x} = \dfrac{25 \ln x}{2x^3 y}$.

36. $(x + xy^2)y' = x^3 y e^{x-y}$.

37. $y' = \cos x (y \csc x - 1)$, $0 < x < \dfrac{\pi}{2}$.

For Problems 38–41, determine which of the five types of differential equations we have studied the given differential equation falls into, and use an appropriate technique to find the solution to the initial-value problem.

38. $y' - x^2 y = x^2$, $y(0) = 5$.

39. $e^{-3x+2y}\, dx + e^{x-4y}\, dy = 0$, $y(0) = 0$.

40. $(3x^2 + 2xy^2)\, dx + (2x^2 y)\, dy = 0$, $y(1) = 3$.

41. $\dfrac{dy}{dx} - (\sin x)y = e^{-\cos x}$, $y(0) = \dfrac{1}{e}$.

42. Determine all values of the constants m and n, if there are any, for which the differential equation

$$(x^5 + y^m)\, dx - x^n y^3\, dy = 0$$

is each of the following:

(a) Exact.

(b) Separable.

(c) Homogeneous.

(d) Linear.

(e) Bernoulli.

43. A man's sandals are moved from poolside (80°F) to a sauna (180°F) to warm and dry them. If they are 100°F after 3 minutes in the sauna, how much time is required in the sauna to increase their temperature to 140°F, according to Newton's law of cooling?

44. A hot plate (150°F) is placed on a countertop in a room kept at 70°F. If the plate cools 25°F in the first 10 minutes, when does the plate reach 100°F, according to Newton's law of cooling?

45. A simple nonlinear law of cooling states that the rate of change of temperature of an object is proportional to the *square* of the temperature difference between the object and its surrounding medium (you may assume that the temperature of the surrounding medium is constant). Set up and solve the initial-value problem that governs this cooling process if the initial temperature is T_0. What happens to the temperature of the object as $t \to \infty$?

46. The temperature of an object at time t is governed by the *linear* differential equation

$$\frac{dT}{dt} = -k(T - 5\cos 2t).$$

At $t = 0$, the temperature of the object is 0°F and is, at that time, increasing at a rate of 5°F/min.

(a) Determine the value of the constant k.

(b) Determine the temperature of the object at time t.

(c) Describe the behavior of the temperature of the object for large values of t.

47. Each spring, sandhill cranes migrate through the Platte River valley in central Nebraska. An estimated maximum of a half-million of these birds reach the region by April 1 each year. If there are only 100,000 sandhill cranes 15 days later and the sandhill cranes leave the Platte River valley at a rate proportional to the number of them still in the valley at the time,

(a) How many sandhill cranes remain in the valley 30 days after April 1?

(b) How many sandhill cranes remain in the valley 35 days after April 1?

(c) How many days after April 1 will there be fewer than 1000 sandhill cranes in the valley?

48. A city's population in the year 2000 was 200,000, in 2003 it was 230,000, and in 2006 it was 250,000. Using the logistic model of population, predict the population in 2010 and 2020.

49. Consider an RC circuit with $R = 4\ \Omega$, $C = \frac{1}{5}$ F, and $E(t) = 6\cos 2t$ V. If $q(0) = 3$ C, determine the current in the circuit for $t \geq 0$.

50. Consider an RL circuit with $R = 3\ \Omega$, $L = 0.3$ H, and $E(t) = 10$ V. If $i(0) = 3$ A, determine the current in the circuit for $t \geq 0$.

51. A solution containing 3 g/L of a salt solution pours into a tank, initially half full of water, at a rate of 6 L/min. The well-stirred mixture flows out at a rate of 4 L/min. If the tank holds 60 L, find the amount of salt (in grams) in the tank when the solution overflows.

In Problems 52–53, use Euler's method with the specified step size to determine the solution to the given initial-value problem at the specified point.

52. $y' = x^2 + 2y^2$, $y(0) = -3$, $h = 0.1$, $y(1)$.

53. $y' = \dfrac{3x}{y} + 2$, $y(1) = 2$, $h = 0.05$, $y(1.5)$.

In Problems 54–55, use the modified Euler method with the specified step size to determine the solution to the given initial-value problem at the specified point. In each case, compare your answer to that determined by using Euler's method.

54. The initial-value problem in Problem 52.

55. The initial-value problem in Problem 53.

In Problems 56–57, use the fourth-order Runge-Kutta method with the specified step size to determine the solution to the given initial-value problem at the specified point. In each case, compare your answer to that determined by using Euler's method.

56. The initial-value problem in Problem 52.

57. The initial-value problem in Problem 53.

Project: A Cylindrical Tank Problem

Consider an open cylindrical tank of height h_0 meters and radius r meters that is filled with water. A circular hole of radius l meters in the bottom of the tank allows the water to flow out under the influence of gravity. According to Torricelli's law, the water flows out with the same speed that it would acquire in falling freely from the water level in the tank to the hole.

1. Use Torricelli's law to derive the following equation for the rate of change of volume of water in the tank,

$$\frac{dV}{dt} = -a\sqrt{2gh}$$

where $h(t)$ denotes the height of water in the tank at time t, a denotes the area of the hole, and g denotes the acceleration due to gravity. [**Hint:** First show that an object that is released from rest at a height h hits the ground with a speed $\sqrt{2gh}$. Then consider the change in the volume of water in the tank in a time interval Δt.]

2. Show that the rate of change of volume of water in the tank is also given by

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}.$$

3. Using the results from problems (1) and (2), determine the height of the water in the tank at time t, and show that the tank will empty when $t = t_e$ where

$$t_e = \frac{\pi r^2}{a}\sqrt{\frac{2h_0}{g}}.$$

4. Suppose now that starting at $t = 0$ chemical is added to the water in the tank at a rate of w grams/second. Derive the following differential equation governing the amount of chemical, $A(t)$, in the tank at time t:

$$\frac{dA}{dt} - \frac{2}{t - t_e}A = w, \quad 0 < t < t_e. \tag{1.12.7}$$

5. Solve the differential equation (1.12.7). Determine the time when $A(t)$ is a maximum.

6. By making an appropriate change of variables in the differential equation (1.12.7), derive a differential equation for the concentration $c(t)$ of chemical in the tank at time t. Solve your differential equation and verify that you get the same expression for $c(t)$ as you do by dividing the expression for $A(t)$ obtained in the previous problem by $V(t)$.

7. In the particular case when $h_0 = 16$ m, $r = 5$ m, $l = 0.1$ m, and $w = 15$ g/s, determine t_e, and the time when the concentration of chemical in the tank reaches 1 g/L.