1.2 Basic Ideas and Terminology

In the preceding section we have used some applied problems to illustrate how differential equations arise. We now undertake to formalize mathematically several ideas introduced through these examples. We begin with a very general definition of a differential equation.

DEFINITION 1.2.1

A **differential equation** is an equation involving one or more derivatives of an unknown function.

Example 1.2.2 The following are all differential equations:

$$\text{(a)}\ \frac{dy}{dx}+y=x^2, \qquad \text{(b)}\ \frac{d^2y}{dx^2}=-k^2y, \qquad \text{(c)}\ \frac{d^3y}{dx^3}+\left(\frac{d^2y}{dx^2}\right)^5+\cos x=0,$$
$$\text{(d)}\ \sin\left(\frac{dy}{dx}\right)+\tan^{-1}y=1, \qquad \text{(e)}\ \phi_{xx}+\phi_{yy}-\phi_x=e^x+x\sin y.$$

The differential equations occurring in (a) through (d) are called **ordinary differential equations**, since the unknown function $y(x)$ depends only on one variable, x. In (e), the unknown function $\phi(x, y)$ depends on more than one variable; hence the equation involves partial derivatives. Such a differential equation is called a **partial differential equation**. In this text we consider only ordinary differential equations. □

We now introduce some more definitions and terminology.

DEFINITION 1.2.3

The order of the highest derivative occurring in a differential equation is called the **order** of the differential equation.

In Example 1.2.2, (a) has order 1, (b) has order 2, (c) has order 3, and (d) has order 1. If we look back at the examples from the previous section, we see that problems formulated using Newton's second law of motion will always be governed by a second-order differential equation (for the position of the object). Indeed, second-order differential equations play a very fundamental role in applied problems, although differential equations of other orders also arise. For example, the differential equation obtained from Newton's law of cooling is a first-order differential equation, as is the differential equation for determining the orthogonal trajectories to a given family of curves. As another example, we note that under certain conditions, the deflection, $y(x)$, of a horizontal beam is governed by the fourth-order differential equation

$$\frac{d^4y}{dx^4}=F(x)$$

for an appropriate function $F(x)$.

Any differential equation of order n can be written in the form

$$G(x, y, y', y'', \ldots, y^{(n)})=0, \tag{1.2.1}$$

where we have introduced the prime notation to denote derivatives, and $y^{(n)}$ denotes the nth derivative of y with respect to x (not y to the power of n). Of particular interest to us throughout the text will be *linear* differential equations. These arise as the special case of Equation (1.2.1), when $y, y', \dots, y^{(n)}$ occur to the first degree only, and not as products or arguments of other functions. The general form for such a differential equation is given in the next definition.

DEFINITION 1.2.4

A differential equation that can be written in the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = F(x),$$

where $a_0, a_1, \dots, a_n$ and F are functions of x only, is called a **linear** differential equation of order n. Such a differential equation is linear in $y, y', y'', \dots, y^{(n)}$.

A differential equation that does not satisfy this definition is called a **nonlinear** differential equation.

Example 1.2.5 The equations

$$y'' + x^2y' + (\sin x)y = e^x \qquad \text{and} \qquad xy''' + 4x^2y' - \frac{2}{1+x^2}y = 0$$

are linear differential equations of order 2 and order 3, respectively, whereas the differential equations

$$y'' + x\sin(y') - xy = x^2 \qquad \text{and} \qquad y'' - x^2y' + y^2 = 0$$

are nonlinear. In the first case the nonlinearity arises from the $\sin(y')$ term, whereas in the second, the nonlinearity is due to the y^2 term. □

Example 1.2.6 The general forms for first- and second-order linear differential equations are

$$a_0(x)\frac{dy}{dx} + a_1(x)y = F(x)$$

and

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x),$$

respectively. □

If we consider the examples from the previous section, we see that the differential equation governing the simple harmonic oscillator is a second-order linear differential equation. In this case the linearity was imposed in the modeling process when we assumed that the restoring force was directly proportional to the displacement from equilibrium (Hooke's law). Not all springs satisfy this relationship. For example, Duffing's equation

$$m\frac{d^2y}{dx^2} + k_1y + k_2y^3 = 0$$

gives a mathematical model of a nonlinear spring–mass system. If $k_2 = 0$, this reduces to the simple harmonic oscillator equation. Newton's law of cooling assumes a linear relationship between the rate of change of the temperature of an object and the temperature difference between the object and that of the surrounding medium. Hence, the resulting differential equation is linear. This can be seen explicitly by writing Equation (1.1.8) as

$$\frac{dT}{dt} + kT = kT_m,$$

which is a first-order linear differential equation. Finally, the differential equation for determining the orthogonal trajectories of a given family of curves will in general be nonlinear, as seen in Example 1.1.1.

Solutions of Differential Equations

We now define precisely what is meant by a solution to a differential equation.

DEFINITION 1.2.7

A function $y = f(x)$ that is (at least) n times differentiable on an interval I is called a **solution** to the differential equation (1.2.1) on I if the substitution $y = f(x)$, $y' = f'(x)$, ... , $y^{(n)} = f^{(n)}(x)$ reduces the differential equation (1.2.1) to an identity valid for all x in I. In this case we say that $y = f(x)$ **satisfies** the differential equation.

Example 1.2.8 Verify that for all constants c_1 and c_2, $y(x) = c_1 \sin x + c_2 \cos x$ is a solution to the linear differential equation $y'' + y = 0$ for x in the interval $(-\infty, \infty)$.

Solution: The function $y(x)$ is certainly twice differentiable for all real x. Furthermore,

$$y'(x) = c_1 \cos x - c_2 \sin x$$

and

$$y''(x) = -(c_1 \sin x + c_2 \cos x).$$

Consequently,

$$y'' + y = -(c_1 \sin x + c_2 \cos x) + c_1 \sin x + c_2 \cos x = 0,$$

so that $y'' + y = 0$ for every x in $(-\infty, \infty)$. It follows from the preceding definition that the given function is a solution to the differential equation on $(-\infty, \infty)$. □

In the preceding example, x could assume all real values. Often, however, the independent variable will be restricted in some manner. For example, the differential equation

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}(y - 1)$$

is undefined when $x \leq 0$, and so any solution would be defined only for $x > 0$. In fact this linear differential equation has solution

$$y(x) = ce^{\sqrt{x}} + 1, \qquad x > 0,$$

where c is a constant. (The reader can check this by plugging in to the given differential equation, as was done in Example 1.2.8. In Section 1.4 we will introduce a technique that will enable us to derive this solution.) We now distinguish two ways in which solutions to a differential equation can be expressed. Often, as in Example 1.2.8, we will be able to obtain a solution to a differential equation in the explicit form $y = f(x)$, for some function f. However, when dealing with nonlinear differential equations, we usually have to be content with a solution written in implicit form

$$F(x, y) = 0,$$

where the function F defines the solution, $y(x)$, implicitly as a function of x. This is illustrated in Example 1.2.9.

Example 1.2.9 Verify that the relation $x^2 + y^2 - 4 = 0$ defines an implicit solution to the nonlinear differential equation

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Solution: We regard the given relation as defining y as a function of x. Differentiating this relation with respect to x yields[4]

$$2x + 2y\frac{dy}{dx} = 0.$$

That is,

$$\frac{dy}{dx} = -\frac{x}{y},$$

as required. In this example we can obtain y explicitly in terms of x, since $x^2+y^2-4 = 0$ implies that

$$y = \pm\sqrt{4 - x^2}.$$

The implicit relation therefore contains the two explicit solutions

$$y(x) = \sqrt{4 - x^2}, \qquad y(x) = -\sqrt{4 - x^2},$$

which correspond graphically to the two semi-circles sketched in Figure 1.2.1.

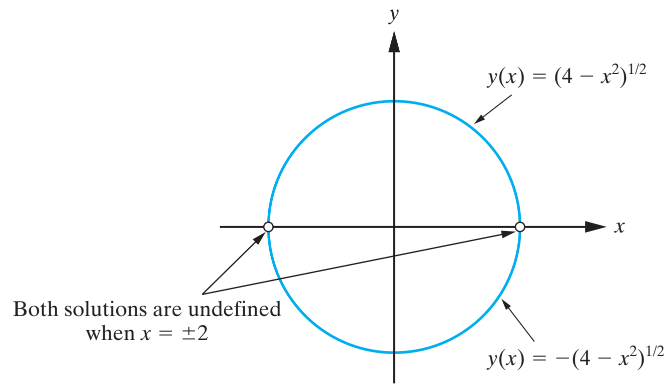

Figure 1.2.1: Two solutions to the differential equation $y' = -x/y$.

[4]Note that we have used implicit differentiation in obtaining $d(y^2)/dx = 2y \cdot (dy/dx)$.

Since $x = \pm 2$ corresponds to $y = 0$ in both of these equations, whereas the differential equation is defined only for $y \neq 0$, we must omit $x = \pm 2$ from the domains of the solutions. Consequently, both of the foregoing solutions to the differential equation are valid for $-2 < x < 2$.

□

In the preceding example the solutions to the differential equation are more simply expressed in implicit form, although, as we have shown, it is quite easy to obtain the corresponding explicit solutions. In the following example the solution must be expressed in implicit form, since it is impossible to solve the implicit relation (analytically) for y as a function of x.

Example 1.2.10 Show that the relation $\sin(xy) + y^2 - x = 0$ defines a solution to

$$\frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy) + 2y}.$$

Solution: Differentiating the given relationship implicitly with respect to x yields

$$\cos(xy)\left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} - 1 = 0.$$

That is,

$$\frac{dy}{dx}[x\cos(xy) + 2y] = 1 - y\cos(xy),$$

which implies that

$$\frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy) + 2y}$$

as required. □

Now consider the simple differential equation

$$\frac{d^2y}{dx^2} = 12x.$$

From elementary calculus we know that all functions whose second derivative is $12x$ can be obtained by performing two integrations. Integrating the given differential equation once yields

$$\frac{dy}{dx} = 6x^2 + c_1,$$

where c_1 is an arbitrary constant. Integrating again, we obtain

$$y(x) = 2x^3 + c_1x + c_2, \tag{1.2.2}$$

where c_2 is another arbitrary constant. The point to notice about this solution is that it contains two arbitrary constants. Further, by assigning appropriate values to these constants, we can determine all solutions to the differential equation. We call (1.2.2) the general solution to the differential equation. In this example the given differential equation was of second-order, and the general solution contained two arbitrary constants, which arose because two integrations were required to solve the differential equation. In the case of an nth-order differential equation we might suspect that the most general form of solution that can arise would contain n arbitrary constants. This is indeed the case and motivates the following definition.

DEFINITION 1.2.11

A solution to an nth-order differential equation on an interval I is called the **general solution on** I if it satisfies the following conditions:

1. The solution contains n constants $c_1, c_2, \ldots, c_n$.
2. All solutions to the differential equation can be obtained by assigning appropriate values to the constants.

Remark Not all differential equations have a general solution. For example, consider

$$(y')^2 + (y-1)^2 = 0.$$

The only solution to this differential equation is $y(x) = 1$, and hence the differential equation does not have a solution containing an arbitrary constant.

Example 1.2.12 Find the general solution to the differential equation $y'' = e^{-x}$.

Solution: Integrating the given differential equation with respect to x yields

$$y' = -e^{-x} + c_1,$$

where c_1 is an integration constant. Integrating this equation, we obtain

$$y(x) = e^{-x} + c_1 x + c_2 \tag{1.2.3}$$

where c_2 is another integration constant. Consequently, all solutions to $y'' = e^{-x}$ are of the form (1.2.3), and therefore, according to Definition 1.2.11, this is the general solution to $y'' = e^{-x}$ on any interval. □

As the preceding example illustrates, we can, in principle, always find the general solution to a differential equation of the form

$$\frac{d^n y}{dx^n} = f(x) \tag{1.2.4}$$

by performing n integrations. However, if the function on the right-hand side of the differential equation is not a function of x only, this procedure cannot be used. Indeed, one of the major aims of this text is to determine solution techniques for differential equations that are more complicated than Equation (1.2.4). A solution to a differential equation is called a **particular solution** if it does not contain any arbitrary constants not present in the differential equation itself. One way in which particular solutions arise is by our assigning specific values to the arbitrary constants occurring in the general solution to a differential equation. For example, from (1.2.3),

$$y(x) = e^{-x} + x$$

is a particular solution to the differential equation $d^2y/dx^2 = e^{-x}$ (the solution corresponding to $c_1 = 1, c_2 = 0$).

Initial-Value Problems

As discussed in the preceding section, the unique specification of an applied problem requires more than just a differential equation. We must also give appropriate auxiliary conditions that characterize the problem under investigation. Of particular interest to us is the case of the initial-value problem defined for an nth-order differential equation as follows.

DEFINITION 1.2.13

An nth-order differential equation together with n auxiliary conditions of the form

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1},$$

where $y_0, y_1, \ldots, y_{n-1}$ are constants, is called an **initial-value problem**.

Example 1.2.14 Solve the initial-value problem

$$y'' = e^{-x}, \tag{1.2.5}$$

$$y(0) = 1, \qquad y'(0) = 4. \tag{1.2.6}$$

Solution: From Example 1.2.12, the general solution to Equation (1.2.5) is

$$y(x) = e^{-x} + c_1 x + c_2. \tag{1.2.7}$$

We now impose the auxiliary conditions (1.2.6). Setting $x = 0$ in (1.2.7), we see that

$$y(0) = 1 \qquad \text{if and only if} \qquad 1 = 1 + c_2.$$

So $c_2 = 0$. Using this value for c_2 in (1.2.7) and differentiating the result yields

$$y'(x) = -e^{-x} + c_1.$$

Consequently

$$y'(0) = 4 \qquad \text{if and only if} \qquad 4 = -1 + c_1,$$

and hence $c_1 = 5$. Thus the given auxiliary conditions pick out the particular solution to the differential equation (1.2.5) with $c_1 = 5$ and $c_2 = 0$, so that the initial-value problem has the unique solution

$$y(x) = e^{-x} + 5x.$$

□

Initial-value problems play a fundamental role in the theory and applications of differential equations. In the previous example, the initial-value problem had a unique solution. More generally, suppose we have a differential equation that can be written in the **normal** form

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}).$$

According to Definition 1.2.13, the initial-value problem for such an nth-order differential equation is the following:

Statement of the initial-value problem: Solve

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$$

subject to

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1},$$

where $y_0, y_1, \ldots, y_{n-1}$ are constants.

It can be shown that this initial-value problem always has a unique solution, provided that f and its partial derivatives with respect to $y, y', \ldots, y^{(n-1)}$ are continuous in an appropriate region. This is a fundamental result in the theory of differential equations. In Chapter 6 we will show how the following special case can be used to develop the theory for *linear* differential equations.

Theorem 1.2.15 Let $a_1, a_2, \ldots, a_n, F$ be functions that are continuous on an interval I. Then, for any x_0 in I, the initial-value problem

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x),$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}$$

has a unique solution on I.

The next example, which we will refer back to on many occasions throughout the remainder of the text, illustrates the power of the preceding theorem.

Example 1.2.16 Prove that the general solution to the differential equation

$$y'' + \omega^2 y = 0, \quad -\infty < x < \infty \tag{1.2.8}$$

where ω is a nonzero constant, is

$$y(x) = c_1 \cos \omega x + c_2 \sin \omega x, \tag{1.2.9}$$

where c_1, c_2 are arbitrary constants.

Solution: It is a routine computation to verify that $y(x) = c_1 \cos \omega x + c_2 \sin \omega x$ is a solution to the differential equation (1.2.8) on $(-\infty, \infty)$. According to Definition 1.2.11 we must now establish that *every* solution to (1.2.8) is of the form (1.2.9). To that end, suppose that $y = f(x)$ is any solution to (1.2.8). Then according to the preceding theorem, $y = f(x)$ is the *unique* solution to the initial-value problem

$$y'' + \omega^2 y = 0, \quad y(0) = f(0), \quad y'(0) = f'(0). \tag{1.2.10}$$

However, consider the function

$$y(x) = f(0) \cos \omega x + \frac{f'(0)}{\omega} \sin \omega x \tag{1.2.11}$$

This is of the form $y(x) = c_1 \cos \omega x + c_2 \sin \omega x$, where $c_1 = f(0)$ and $c_2 = f'(0)/\omega$, and therefore solves the differential equation (1.2.8). Further, evaluating (1.2.11) at $x = 0$ yields

$$y(0) = f(0) \quad \text{and} \quad y'(0) = f'(0).$$

Consequently, (1.2.11) solves the initial-value problem (1.2.10). But, by assumption, $y(x) = f(x)$ solves the same initial-value problem. Owing to the uniqueness of the solution to this initial-value problem, it follows that these two solutions must coincide. Therefore,

$$f(x) = f(0)\cos\omega x + \frac{f'(0)}{\omega}\sin\omega x = c_1\cos\omega x + c_2\sin\omega x.$$

Since $f(x)$ was an arbitrary solution to the differential equation (1.2.8), we can conclude that every solution to (1.2.8) is of the form

$$y(x) = c_1\cos\omega x + c_2\sin\omega x$$

and therefore this is the general solution on $(-\infty, \infty)$. □

In the remainder of this chapter we will focus primarily on first-order differential equations and some of their elementary applications. We will investigate such differential equations qualitatively, analytically, and numerically.

Exercises for 1.2

Key Terms

Differential equation, Order of a differential equation, Linear differential equation, Nonlinear differential equation, General solution to a differential equation, Particular solution to a differential equation, Initial-value problem.

Skills

- Be able to determine the order of a differential equation.
- Be able to determine whether a given differential equation is linear or nonlinear.
- Be able to determine whether or not a given function $y(x)$ is a particular solution to a given differential equation.
- Be able to determine whether or not a given implicit relation defines a particular solution to a given differential equation.
- Be able to find the general solution to differential equations of the form $y^{(n)} = f(x)$ via n integrations.
- Be able to use initial conditions to find the solution to an initial-value problem.

True-False Review

For Questions 1–6, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. The order of a differential equation is the order of the lowest derivative appearing in the differential equation.

2. The general solution to a third-order differential equation must contain three constants.

3. An initial-value problem always has a unique solution if the functions and partial derivatives involved are continuous.

4. The general solution to $y'' + y = 0$ is $y(x) = c_1\cos x + 5c_2\cos x$.

5. The general solution to $y'' + y = 0$ is $y(x) = c_1\cos x + 5c_1\sin x$.

6. The general solution to a differential equation of the form $y^{(n)} = F(x)$ can be obtained by n consecutive integrations of the function $F(x)$.

Problems

For Problems 1–6, determine the order of the given differential equation and state whether it is linear or nonlinear.

1. $\dfrac{d^2y}{dx^2} + e^{xy}\dfrac{dy}{dx} = x^2.$

2. $\dfrac{d^3y}{dx^3} + 4\dfrac{d^2y}{dx^2} + \sin x\dfrac{dy}{dx} = xy + \tan x$.

3. $y'' + 3x(y')^3 - y = 1 + 3x$.

4. $\sin x \cdot e^{y''} + y' - \tan y = \cos x$.

5. $\dfrac{d^4y}{dx^4} + 3\dfrac{d^2y}{dx^2} = x$.

6. $\sqrt{x}y'' + \dfrac{\ln x}{y'''} = 3x^3$.

For Problems 7–18, verify that the given function is a solution to the given differential equation (c_1 and c_2 are arbitrary constants), and state the maximum interval over which the solution is valid.

7. $y(x) = c_1e^x\cos 2x + c_2e^x\sin 2x, \quad y'' - 2y' + 5y = 0$.

8. $y(x) = c_1e^x + c_2e^{-2x}, \quad y'' + y' - 2y = 0$.

9. $y(x) = \dfrac{1}{x+4}, \quad y' = -y^2$.

10. $y(x) = c_1x^{1/2}, \quad y' = \dfrac{y}{2x}$.

11. $y(x) = e^{-x}\sin 2x, \quad y'' + 2y' + 5y = 0$.

12. $y(x) = c_1\cosh 3x + c_2\sinh 3x, \quad y'' - 9y = 0$.

13. $y(x) = c_1x^{-3} + c_2x^{-1}, \quad x^2y'' + 5xy' + 3y = 0$.

14. $y(x) = c_1x^{1/2} + 3x^2, \quad 2x^2y'' - xy' + y = 9x^2$.

15. $y(x) = c_1x^2 + c_2x^3 - x^2\sin x$,
$x^2y'' - 4xy' + 6y = x^4\sin x$.

16. $y(x) = c_1e^{ax} + c_2e^{bx}, \quad y'' - (a+b)y' + aby = 0$, where a and b are constants and $a \neq b$.

17. $y(x) = e^{ax}(c_1 + c_2x), \quad y'' - 2ay' + a^2y = 0$, where a is a constant.

18. $y(x) = e^{ax}(c_1\cos bx + c_2\sin bx)$,
$y'' - 2ay' + (a^2 + b^2)y = 0$, where a and b are constants.

For Problems 19–22, determine all values of the constant r such that the given function solves the given differential equation.

19. $y(x) = e^{rx}, \quad y'' + 2y' - 3y = 0$.

20. $y(x) = e^{rx}, \quad y'' - 8y' + 16y = 0$.

21. $y(x) = x^r, \quad x^2y'' + xy' - y = 0$.

22. $y(x) = x^r, \quad x^2y'' + 5xy' + 4y = 0$.

23. When N is a positive integer, the **Legendre equation**

$$(1 - x^2)y'' - 2xy' + N(N+1)y = 0,$$

with $-1 < x < 1$, has a solution that is a polynomial of degree N. Show by substitution into the differential equation that in the case $N = 3$ such a solution is

$$y(x) = \frac{1}{2}x(5x^2 - 3).$$

24. Determine a solution to the differential equation

$$(1 - x^2)y'' - xy' + 4y = 0$$

of the form $y(x) = a_0 + a_1x + a_2x^2$ satisfying the normalization condition $y(1) = 1$.

For Problems 25–29, show that the given relation defines an implicit solution to the given differential equation, where c is an arbitrary constant.

25. $x\sin y - e^x = c, \quad y' = \dfrac{e^x - \sin y}{x\cos y}$.

26. $xy^2 + 2y - x = c, \quad y' = \dfrac{1 - y^2}{2(1 + xy)}$.

27. $e^{xy} - x = c, \quad y' = \dfrac{1 - ye^{xy}}{xe^{xy}}$.
Determine the solution with $y(1) = 0$.

28. $e^{y/x} + xy^2 - x = c, \quad y' = \dfrac{x^2(1 - y^2) + ye^{y/x}}{x(e^{y/x} + 2x^2y)}$.

29. $x^2y^2 - \sin x = c, \quad y' = \dfrac{\cos x - 2xy^2}{2x^2y}$.
Determine the explicit solution that satisfies $y(\pi) = 1/\pi$.

For Problems 30–33, find the general solution to the given differential equation and the maximum interval on which the solution is valid.

30. $y' = \sin x$.

31. $y' = x^{-1/2}$.

32. $y'' = xe^x$.

33. $y'' = x^n$, n an integer.

For Problems 34–38, solve the given initial-value problem.

34. $y' = \ln x, \quad y(1) = 2$.

35. $y'' = \cos x, \quad y(0) = 2, \quad y'(0) = 1.$

36. $y''' = 6x, \quad y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 4.$

37. $y'' = xe^x, \quad y(0) = 3, \quad y'(0) = 4.$

38. Prove that the general solution to $y'' - y = 0$ on any interval I is $y(x) = c_1 e^x + c_2 e^{-x}$.

A second-order differential equation together with two auxiliary conditions imposed at different values of the independent variable is called a **boundary-value problem**. For Problems 39–40, solve the given boundary-value problem.

39. $y'' = e^{-x}, \quad y(0) = 1, \quad y(1) = 0.$

40. $y'' = -2(3 + 2\ln x), \quad y(1) = y(e) = 0.$

41. The differential equation $y'' + y = 0$ has the general solution $y(x) = c_1 \cos x + c_2 \sin x$.

(a) Show that the boundary-value problem $y'' + y = 0, \quad y(0) = 0, \quad y(\pi) = 1$ has no solutions.

(b) Show that the boundary-value problem $y'' + y = 0, \quad y(0) = 0, \quad y(\pi) = 0$, has an infinite number of solutions.

For Problems 42–47, verify that the given function is a solution to the given differential equation. In these problems, c_1 and c_2 are arbitrary constants. Throughout the text, the symbol $\diamond$ refers to exercises for which some form of technology, such as a graphing calculator or computer algebra system (CAS), is recommended.

42. $\diamond$ $y(x) = c_1 e^{2x} + c_2 e^{-3x}, \quad y'' + y' - 6y = 0.$

43. $\diamond$ $y(x) = c_1 x^4 + c_2 x^{-2}, \quad x^2 y'' - xy' - 8y = 0, x > 0.$

44. $\diamond$ $y(x) = c_1 x^2 + c_2 x^2 \ln x + \frac{1}{6} x^2 (\ln x)^3$, $x^2 y'' - 3xy' + 4y = x^2 \ln x, \quad x > 0.$

45. $\diamond$ $y(x) = x^a [c_1 \cos(b \ln x) + c_2 \sin(b \ln x)]$, $x^2 y'' + (1 - 2a)xy' + (a^2 + b^2)y = 0, \quad x > 0$, where a and b are arbitrary constants.

46. $\diamond$ $y(x) = c_1 e^x + c_2 e^{-x}(1 + 2x + 2x^2)$, $xy'' - 2y' + (2 - x)y = 0, \quad x > 0.$

47. $\diamond$ $y(x) = \sum_{k=0}^{10} \frac{1}{k!} x^k, \quad xy'' - (x + 10)y' + 10y = 0,$ $x > 0.$

48. $\diamond$

(a) Derive the polynomial of degree five that satisfies both the Legendre equation

$$(1 - x^2)y'' - 2xy' + 30y = 0$$

and the normalization condition $y(1) = 1$.

(b) $\diamond$ Sketch your solution from (a) and determine approximations to all zeros and local maxima and local minima on the interval $(-1, 1)$.

49. $\diamond$ One solution to the Bessel equation of (nonnegative) integer order N

$$x^2 y'' + xy' + (x^2 - N^2)y = 0$$

is

$$y(x) = J_N(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(N + k)!} \left(\frac{x}{2}\right)^{2k+N}.$$

(a) Write the first three terms of $J_0(x)$.

(b) Let $J(0, x, m)$ denote the mth partial sum

$$J(0, x, m) = \sum_{k=0}^{m} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}.$$

Plot $J(0, x, 4)$ and use your plot to approximate the first positive zero of $J_0(x)$. Compare your value against a tabulated value or one generated by a computer algebra system.

(c) Plot $J_0(x)$ and $J(0, x, 4)$ on the same axes over the interval $[0, 2]$. How well do they compare?

(d) If your system has built-in Bessel functions, plot $J_0(x)$ and $J(0, x, m)$ on the same axes over the interval $[0, 10]$ for various values of m. What is the smallest value of m that gives an accurate approximation to the first *three* positive zeros of $J_0(x)$?

1.3 The Geometry of First-Order DIfferential Equations

The primary aim of this chapter is to study the first-order differential equation

$$\frac{dy}{dx} = f(x, y), \tag{1.3.1}$$