35. $y'' = \cos x, \quad y(0) = 2, \quad y'(0) = 1.$

36. $y''' = 6x, \quad y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 4.$

37. $y'' = xe^x, \quad y(0) = 3, \quad y'(0) = 4.$

38. Prove that the general solution to $y'' - y = 0$ on any interval I is $y(x) = c_1e^x + c_2e^{-x}$.

A second-order differential equation together with two auxiliary conditions imposed at different values of the independent variable is called a **boundary-value problem**. For Problems 39–40, solve the given boundary-value problem.

39. $y'' = e^{-x}, \quad y(0) = 1, \quad y(1) = 0.$

40. $y'' = -2(3 + 2\ln x), \quad y(1) = y(e) = 0.$

41. The differential equation $y'' + y = 0$ has the general solution $y(x) = c_1 \cos x + c_2 \sin x$.

- **(a)** Show that the boundary-value problem $y'' + y = 0, \quad y(0) = 0, \quad y(\pi) = 1$ has no solutions.
- **(b)** Show that the boundary-value problem $y'' + y = 0, \quad y(0) = 0, \quad y(\pi) = 0$, has an infinite number of solutions.

For Problems 42–47, verify that the given function is a solution to the given differential equation. In these problems, c_1 and c_2 are arbitrary constants. Throughout the text, the symbol ◇ refers to exercises for which some form of technology, such as a graphing calculator or computer algebra system (CAS), is recommended.

42. ◇ $y(x) = c_1e^{2x} + c_2e^{-3x}, \quad y'' + y' - 6y = 0.$

43. ◇ $y(x) = c_1x^4 + c_2x^{-2}, \quad x^2y'' - xy' - 8y = 0, x > 0.$

44. ◇ $y(x) = c_1x^2 + c_2x^2 \ln x + \frac{1}{6}x^2(\ln x)^3$, $x^2y'' - 3xy' + 4y = x^2 \ln x, \quad x > 0.$

45. ◇ $y(x) = x^a[c_1 \cos(b \ln x) + c_2 \sin(b \ln x)]$, $x^2y'' + (1 - 2a)xy' + (a^2 + b^2)y = 0, \quad x > 0$, where a and b are arbitrary constants.

46. ◇ $y(x) = c_1e^x + c_2e^{-x}(1 + 2x + 2x^2)$, $xy'' - 2y' + (2 - x)y = 0, \quad x > 0.$

47. ◇ $y(x) = \sum_{k=0}^{10} \frac{1}{k!}x^k, \quad xy'' - (x + 10)y' + 10y = 0$, $x > 0.$

48. ◇

- **(a)** Derive the polynomial of degree five that satisfies both the Legendre equation

$$(1 - x^2)y'' - 2xy' + 30y = 0$$

and the normalization condition $y(1) = 1$.

- **(b)** ◇ Sketch your solution from (a) and determine approximations to all zeros and local maxima and local minima on the interval $(-1, 1)$.

49. ◇ One solution to the Bessel equation of (nonnegative) integer order N

$$x^2y'' + xy' + (x^2 - N^2)y = 0$$

is

$$y(x) = J_N(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(N + k)!} \left(\frac{x}{2}\right)^{2k+N}.$$

- **(a)** Write the first three terms of $J_0(x)$.
- **(b)** Let $J(0, x, m)$ denote the mth partial sum

$$J(0, x, m) = \sum_{k=0}^{m} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}.$$

Plot $J(0, x, 4)$ and use your plot to approximate the first positive zero of $J_0(x)$. Compare your value against a tabulated value or one generated by a computer algebra system.

- **(c)** Plot $J_0(x)$ and $J(0, x, 4)$ on the same axes over the interval [0, 2]. How well do they compare?
- **(d)** If your system has built-in Bessel functions, plot $J_0(x)$ and $J(0, x, m)$ on the same axes over the interval [0, 10] for various values of m. What is the smallest value of m that gives an accurate approximation to the first *three* positive zeros of $J_0(x)$?

1.3 The Geometry of First-Order DIfferential Equations

The primary aim of this chapter is to study the first-order differential equation

$$\frac{dy}{dx} = f(x, y), \tag{1.3.1}$$

where $f(x, y)$ is a given function of x and y. In this section we focus our attention mainly on the geometric aspects of the differential equation and its solutions. The graph of any solution to the differential equation (1.3.1) is called a **solution curve**. If we recall the geometric interpretation of the derivative dy/dx as giving the slope of the tangent line at any point on the curve with equation $y = y(x)$, we see that the function $f(x, y)$ in (1.3.1) gives the slope of the tangent line to the solution curve passing through the point (x, y). Consequently, when we solve Equation (1.3.1), we are finding all curves whose slope at the point (x, y) is given by the function $f(x, y)$. According to our definition in the previous section, the general solution to the differential equation (1.3.1) will involve one arbitrary constant, and therefore, geometrically, the general solution gives a family of solution curves in the xy-plane, one solution curve corresponding to each value of the arbitrary constant.

Example 1.3.1 Find the general solution to the differential equation $dy/dx = 2x$, and sketch the corresponding solution curves.

Solution: The differential equation can be integrated directly to obtain $y(x) = x^2+c$. Consequently the solution curves are a family of parabolas in the xy-plane. This is illustrated in Figure 1.3.1. □

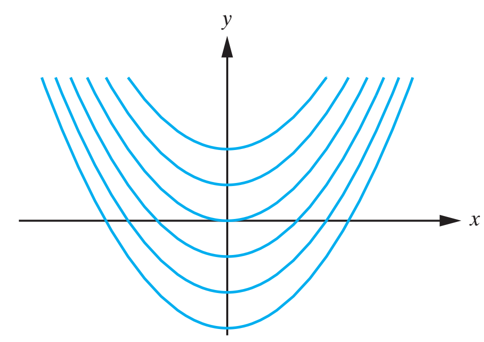

Figure 1.3.1: Some solution curves for the differential equation $dy/dx = 2x$.

Figure 1.3.2 gives a Mathematica plot of some solution curves to the differential equation

$$\frac{dy}{dx} = y - x^2.$$

This illustrates that generally the solution curves of a differential equation are quite complicated. Upon completion of the material in this section, the reader will be able to obtain Figure 1.3.2 without needing a computer algebra system.

Existence and Uniqueness of Solutions

It is useful for the further analysis of the differential equation (1.3.1) to give at this point a brief discussion of the existence and uniqueness of solutions to the corresponding initial-value problem

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0. \tag{1.3.2}$$

Geometrically, we are interested in finding the particular solution curve to the differential equation that passes through the point in the xy-plane with coordinates (x_0, y_0). The following questions arise regarding the initial-value problem:

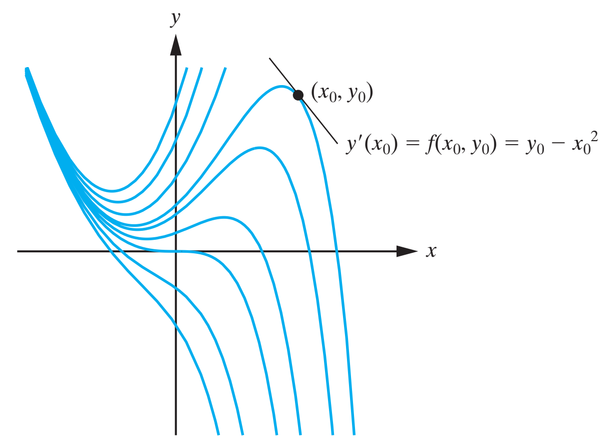

Figure 1.3.2: Some solution curves for the differential equation $dy/dx = y - x^2$.

1. Existence: Does the initial-value problem have any solutions?
2. Uniqueness: If the answer to question 1 is yes, does the initial-value problem have only one solution?

Certainly in the case of an applied problem we would be interested only in initial-value problems that have precisely one solution. The following theorem establishes conditions on f that guarantee the existence and uniqueness of a solution to the initial-value problem (1.3.2).

Theorem 1.3.2 **(Existence and Uniqueness Theorem)**

Let $f(x, y)$ be a function that is continuous on the rectangle

$$R = \{(x, y) : a \le x \le b, c \le y \le d\}.$$

Suppose further that $\partial f/\partial y$ is continuous in R. Then for any interior point (x_0, y_0) in the rectangle R, there exists an interval I containing x_0 such that the initial-value problem (1.3.2) has a unique solution for x in I.

Proof A complete proof of this theorem can be found, for example, in G. F. Simmons, *Differential Equations* (New York: McGraw-Hill, 1972). Figure 1.3.3 gives a geometric illustration of the result. ■

Remark From a geometric viewpoint, if $f(x, y)$ satisfies the hypotheses of the existence and uniqueness theorem in a region R of the xy-plane, then throughout that region the solution curves of the differential equation $dy/dx = f(x, y)$ cannot intersect. For if two solution curves did intersect at (x_0, y_0) in R, then that would imply there was more than one solution to the initial-value problem

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0,$$

which would contradict the existence and uniqueness theorem.

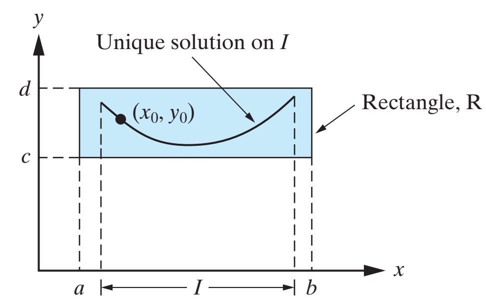

Figure 1.3.3: Illustration of the existence and uniqueness theorem for first-order differential equations.

The following example illustrates how the preceding theorem can be used to establish the existence of a unique solution to a differential equation, even though at present we do not know how to determine the solution.

Example 1.3.3 Prove that the initial-value problem

$$\frac{dy}{dx} = 3xy^{1/3}, \qquad y(0) = a$$

has a unique solution whenever $a \neq 0$.

Solution: In this case the initial point is $x_0 = 0$, $y_0 = a$, and $f(x, y) = 3xy^{1/3}$. Hence, $\partial f/\partial y = xy^{-2/3}$. Consequently, f is continuous at all points in the xy-plane, whereas $\partial f/\partial y$ is continuous at all points not lying on the x-axis ($y \neq 0$). Provided $a \neq 0$, we can certainly draw a rectangle containing $(0, a)$ that does not intersect the x-axis. (See Figure 1.3.4.) In any such rectangle the hypotheses of the existence and uniqueness theorem are satisfied, and therefore the initial-value problem does indeed have a unique solution. □

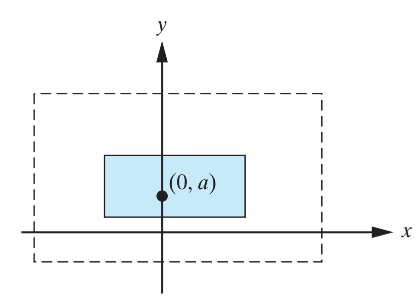

Figure 1.3.4: The initial-value problem in Example 1.3.3 satisfies the hypotheses of the existence and uniqueness theorem in the small rectangle, but not in the large rectangle.

Example 1.3.4 Discuss the existence and uniqueness of solutions to the initial-value problem

$$\frac{dy}{dx} = 3xy^{1/3}, \qquad y(0) = 0.$$

Solution: The differential equation is the same as in the previous example, but the initial condition is imposed on the x-axis. Since $\partial f/\partial y = xy^{-2/3}$ is not continuous along the x-axis, there is no rectangle containing (0, 0) in which the hypotheses of the existence and uniqueness theorem are satisfied. *We can therefore draw no conclusion from the theorem itself.* We leave it as an exercise to verify by direct substitution that the given initial-value problem does in fact have the following two solutions:

$$y(x) = 0 \quad \text{and} \quad y(x) = x^3.$$

Consequently in this case the initial-value problem does not have a unique solution. □

Slope Fields

We now return to our discussion of the geometry of solutions to the differential equation

$$\frac{dy}{dx} = f(x, y).$$

The fact that the function $f(x, y)$ gives the slope of the tangent line to the solution curves of this differential equation leads to a simple and important idea for determining the overall shape of the solution curves. We compute the value of $f(x, y)$ at several points and draw through each of the corresponding points in the xy-plane small line segments having $f(x, y)$ as their slopes. The resulting sketch is called the **slope field** for the differential equation. The key point is that each solution curve must be tangent to the line segments that we have drawn, and therefore by studying the slope field we can obtain the general shape of the solution curves.

Example 1.3.5 Sketch the slope field for the differential equation $dy/dx = 2x^2$.

Solution: The slope of the solution curves to the differential equation at each point in the xy-plane depends on x only. Consequently, the slopes of the solution curves will be the same at every point on any line parallel to the y-axis (on such a line, x is constant). Table 1.3.1 contains the values of the slope of the solution curves at various points in the interval [−1, 1].

x	Slope $= 2x^2$
0	0
±0.2	0.08
±0.4	0.32
±0.6	0.72
±0.8	1.28
±1.0	2

Table 1.3.1: Values of the slope for the differential equation in Example 1.3.5.

Using this information, we obtain the slope field shown in Figure 1.3.5. In this example, we can integrate the differential equation to obtain the general solution

$$y(x) = \frac{2}{3}x^3 + c.$$

Some solution curves and their relation to the slope field are also shown in Figure 1.3.5. □

In the preceding example, the slope field could be obtained fairly easily because the slopes of the solution curves to the differential equation were constant on lines parallel to the y-axis. For more complicated differential equations, further analysis is generally required if we wish to obtain an accurate plot of the slope field and the behavior of the corresponding solution curves. Below we have listed three useful procedures.

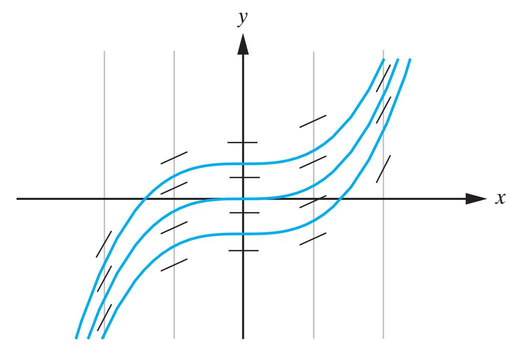

Figure 1.3.5: Slope field and some representative solution curves for the differential equation $dy/dx = 2x^2$.

1. *Isoclines*: For the differential equation

$$\frac{dy}{dx} = f(x, y), \tag{1.3.3}$$

the function $f(x, y)$ determines the regions in the xy-plane where the slope of the solution curves is positive, as well as those where it is negative. Furthermore, each solution curve will have the same slope k along the family of curves

$$f(x, y) = k.$$

These curves are called the **isoclines** of the differential equation, and they can be very useful in determining slope fields. When sketching a slope field, we often start by drawing several isoclines and the corresponding line segments with slope k at various points along them.

2. *Equilibrium Solutions*: Any solution to the differential equation (1.3.3) of the form $y(x) = y_0$, where y_0 is a constant, is called an **equilibrium solution** to the differential equation. The corresponding solution curve is a line parallel to the x-axis. From Equation (1.3.3), equilibrium solutions are given by any constant values of y for which $f(x, y) = 0$, and therefore can often be obtained by inspection. For example, the differential equation

$$\frac{dy}{dx} = (y - x)(y + 1)$$

has the equilibrium solution $y(x) = -1$. One reason that equilibrium solutions are useful in sketching slope fields and determining the general behavior of the full family of solution curves is that, from the existence and uniqueness theorem, we know that no other solution curves can intersect the solution curve corresponding to an equilibrium solution. Consequently, equilibrium solutions serve to divide the xy-plane into different regions.

3. *Concavity Changes*: By differentiating Equation (1.3.3) (implicitly) with respect to x we can obtain an expression for d^2y/dx^2 in terms of x and y. This can be useful in determining the behavior of the concavity of the solution curves to the differential equation (1.3.3). The remaining examples illustrate the application of the foregoing procedures.

Example 1.3.6 Sketch the slope field for the differential equation

$$\frac{dy}{dx} = y - x. \tag{1.3.4}$$

Solution: By inspection we see that the differential equation has no equilibrium solutions. The isoclines of the differential equation are the family of straight lines $y-x=k$. Thus each solution curve of the differential equation has slope k at all points along the line $y-x=k$. Table 1.3.2 contains several values for the slopes of the solution curves, and the equations of the corresponding isoclines. We note that the slope at all points along the isocline $y=x+1$ is unity, which, from Table 1.3.2, coincides with the slope of any solution curve that meets it. This implies that the isocline must in fact coincide with a solution curve. Hence, one solution to the differential equation (1.3.4) is $y(x)=x+1$, and, by the existence and uniqueness theorem, no other solution curve can intersect this one.

Slope of Solution Curves	Equation of Isocline
$k=-2$	$y=x-2$
$k=-1$	$y=x-1$
$k=0$	$y=x$
$k=1$	$y=x+1$
$k=2$	$y=x+2$

Table 1.3.2: Slope and isocline information for the differential equation in Example 1.3.6.

In order to determine the behavior of the concavity of the solution curves, we differentiate the given differential equation implicitly with respect to x to obtain

$$\frac{d^2y}{dx^2}=\frac{dy}{dx}-1=y-x-1,$$

where we have used (1.3.4) to substitute for dy/dx in the second step. We see that the solution curves are concave up ($y''>0$) at all points above the line

$$y=x+1 \tag{1.3.5}$$

and concave down ($y''<0$) at all points beneath this line. We also note that Equation (1.3.5) coincides with the particular solution already identified. Putting all of this information together, we obtain the slope field sketched in Figure 1.3.6.

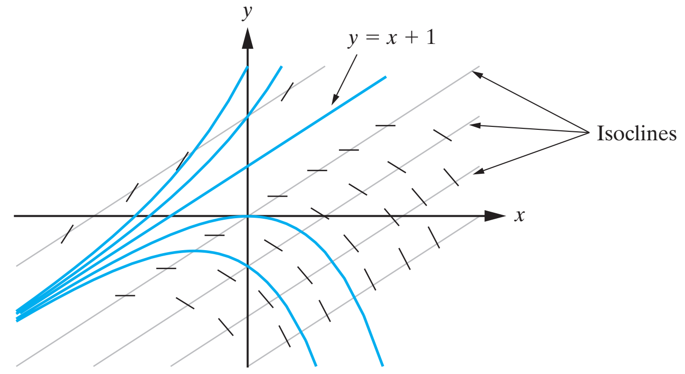

Figure 1.3.6: Hand-drawn slope field, isoclines, and some approximate solution curves for the differential equation in Example 1.3.6.

□

Generating Slope Fields Using Technology

Many computer algebra systems (CAS) and graphing calculators have built-in programs to generate slope fields. As an example, in the CAS Maple the command

$$\text{diffeq} := \text{diff}(y(x), x) = y(x) - x;$$

assigns the name diffeq to the differential equation considered in the previous example. The further command

$$\text{DEplot}(\text{diffeq}, y(x), x = -3..3, y = -3..3, \text{arrows=line});$$

then produces a sketch of the slope field for the differential equation on the square $-3 \leq x \leq 3, -3 \leq y \leq 3$. Initial conditions such as $y(0) = 0$, $y(0) = 1$, $y(0) = 2$, $y(0) = -1$ can be specified using the command

$$\text{IC} := \{[0, 0], [0, 1], [0, 2], [0, -1]\};$$

Then the command

$$\text{DEplot}(\text{diffeq}, y(x), x = -3..3, \text{IC}, y = -3..3, \text{arrows=line});$$

not only plots the slope field, but also gives a numerical approximation to each of the solution curves satisfying the specified initial conditions. Some of the methods that can be used to generate such numerical approximations will be discussed in Section 1.10. The preceding sequence of Maple commands was used to generate the Maple plot given in Figure 1.3.7. Clearly the generation of slope fields and approximate solution curves is one area where technology can be extremely helpful.

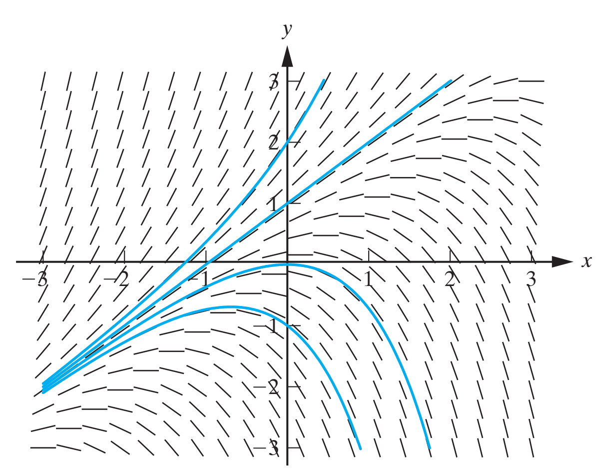

Figure 1.3.7: Maple plot of the slope field and some approximate solution curves for the differential equation in Example 1.3.6.

Example 1.3.7 Sketch the slope field and some approximate solution curves for the differential equation

$$\frac{dy}{dx} = y(2 - y). \tag{1.3.6}$$

Solution: We first note that the given differential equation has the two equilibrium solutions

$$y(x) = 0 \quad \text{and} \quad y(x) = 2.$$

Consequently, from Theorem 1.3.2, the xy-plane can be divided into the three distinct regions $y < 0$, $0 < y < 2$, and $y > 2$. From Equation (1.3.6) the behavior of the sign of the slope of the solution curves in each of these regions is given in the following schematic.

sign of slope: − − −− |+ + ++ |− − −−
y-interval: 0 2

The isoclines are determined from

$$y(2 - y) = k.$$

That is,

$$y^2 - 2y + k = 0,$$

so that the solution curves have slope k at all points of intersection with the horizontal lines

$$y = 1 \pm \sqrt{1 - k}. \tag{1.3.7}$$

Table 1.3.3 contains some of the isocline equations. Note from Equation (1.3.7) that the largest possible positive slope is $k = 1$. We see that the slopes of the solution curves quickly become very large and negative for y outside the interval [0, 2]. Finally, differentiating Equation (1.3.6) implicitly with respect to x yields

$$\frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 2y\frac{dy}{dx} = 2(1 - y)\frac{dy}{dx} = 2y(1 - y)(2 - y).$$

Slope of Solution Curves	**Equation of Isocline**
$k = 1$	$y = 1$
$k = 0$	$y = 2$ and $y = 0$
$k = -1$	$y = 1 \pm \sqrt{2}$
$k = -2$	$y = 1 \pm \sqrt{3}$
$k = -3$	$y = 3$ and $y = -1$
$k = -n, n \geq 1$	$y = 1 \pm \sqrt{n + 1}$

Table 1.3.3: Slope and isocline information for the differential equation in Example 1.3.7.

The sign of d^2y/dx^2 is given in the following schematic.

sign of y'': − − −− |+ + ++ |− − −− |+ + + +
y-interval: 0 1 2

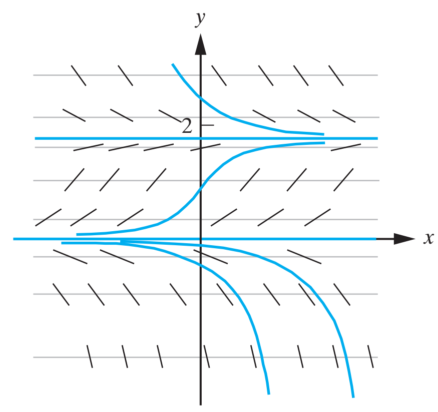

Figure 1.3.8: Hand-drawn slope field, isoclines, and some solution curves for the differential equation $dy/dx = y(2 - y)$.

Using this information leads to the slope field sketched in Figure 1.3.8. We have also included some approximate solution curves. We see from the slope field that for any initial condition $y(x_0) = y_0$, with $0 \leq y_0 \leq 2$, the corresponding unique solution to the differential equation will be bounded. In contrast, if $y_0 > 2$, the slope field suggests that all corresponding solutions approach $y = 2$ as $x \to \infty$, whereas if $y_0 < 0$, then all corresponding solutions approach $y = 0$ as $x \to -\infty$. Furthermore, the behavior of the slope field also suggests that the solution curves that do not lie in the region $0 < y < 2$ may diverge at finite values of x. We leave it as an exercise to verify (by substitution into Equation (1.3.6)) that for all values of the constant c,

$$y(x) = \frac{2ce^{2x}}{ce^{2x} - 1}$$

is a solution to the given differential equation. We see that any initial condition that yields a positive value for c will indeed lead to a solution that has a vertical asymptote at $x = \frac{1}{2}\ln(1/c)$. □

The tools that we have introduced in this section enable us to analyze the solution behavior of many first-order differential equations. However, for complicated functions $f(x, y)$ in Equation (1.3.3), performing these computations by hand can be a tedious task. Fortunately, as we have illustrated, there are many computer programs available for drawing slope fields and generating solution curves (numerically). Furthermore, several graphing calculators also have these capabilities.

Exercises for 1.3

Key Terms

Solution curve, Existence and uniqueness theorem, Slope field, Isocline, Equilibrium solution.

Skills

- Be able to find isoclines for a differential equation $dy/dx = f(x, y)$.
- Be able to determine equilibrium solutions for a differential equation $dy/dx = f(x, y)$.
- Be able to sketch the slope field for a differential equation, using isoclines, equilibrium solutions, and concavity changes.
- Be able to sketch solution curves to a differential equation.
- Be able to apply the existence and uniqueness theorem to find unique solutions to initial-value problems.

True-False Review

For Questions 1–7, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. If $f(x, y)$ satisfies the hypotheses of the existence and uniqueness theorem in a region R of the xy-plane, then the solution curves to a differential equation $dy/dx = f(x, y)$ cannot intersect in R.

2. Every differential equation $dy/dx = f(x, y)$ has at least one equilibrium solution.

3. The differential equation $dy/dx = x(y^2 - 4)$ has no equilibrium solutions.

4. The circle $x^2+y^2 = 4$ is an isocline for the differential equation $dy/dx = x^2 + y^2$.

5. The equilibrium solutions of a differential equation are always parallel to one another.

6. The isoclines for the differential equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2y}$$

are the family of circles $x^2 + (y - k)^2 = k^2$.

7. No solution to the differential equation $dy/dx = f(x, y)$ can intersect with equilibrium solutions of the differential equation.

Problems

For Problems 1–7, determine the differential equation giving the slope of the tangent line at the point (x, y) for the given family of curves.

1. $y = c/x$.

2. $y = cx^2$.

3. $x^2 + y^2 = 2cx$.

4. $y^2 = cx$.

5. $2cy = x^2 - c^2$.

6. $y^2 - x^2 = c$.

7. $(x - c)^2 + (y - c)^2 = 2c^2$.

For Problems 8–11, verify that the given function (or relation) defines a solution to the given differential equation and sketch some of the solution curves. If an initial condition is given, label the solution curve corresponding to the resulting unique solution. (In these problems, c denotes an arbitrary constant.)

8. $x^2 + y^2 = c, \quad y' = -x/y$.

9. $y = cx^3, \quad y' = 3y/x, \quad y(2) = 8$.

10. $y^2 = cx, \quad 2x\,dy - y\,dx = 0, \quad y(1) = 2$.

11. $(x - c)^2 + y^2 = c^2, \quad y' = \dfrac{y^2 - x^2}{2xy}, \quad y(2) = 2$.

12. Prove that the initial-value problem

$$y' = x \sin(x + y), \quad y(0) = 1$$

has a unique solution.

13. Use the existence and uniqueness theorem to prove that $y(x) = 3$ is the only solution to the initial-value problem

$$y' = \frac{x}{x^2 + 1}(y^2 - 9), \quad y(0) = 3.$$

14. Do you think that the initial-value problem

$$y' = xy^{1/2}, \quad y(0) = 0$$

has a unique solution? Justify your answer.

15. Even simple-looking differential equations can have complicated solution curves. In this problem, we study the solution curves of the differential equation

$$y' = -2xy^2. \qquad (1.3.8)$$

(a) Verify that the hypotheses of the existence and uniqueness theorem (Theorem 1.3.2) are satisfied for the initial-value problem

$$y' = -2xy^2, \quad y(x_0) = y_0$$

for every (x_0, y_0). This establishes that the initial-value problem always has a unique solution on some interval containing x_0.

(b) Verify that for all values of the constant c, $y(x) = 1/(x^2 + c)$ is a solution to (1.3.8).

(c) Use the solution to (1.3.8) given in (b) to solve the following initial-value problems. For each case, sketch the corresponding solution curve, and state the maximum interval on which your solution is valid.

(i) $y' = -2xy^2, \quad y(0) = 1.$
(ii) $y' = -2xy^2, \quad y(1) = 1.$
(iii) $y' = -2xy^2, \quad y(0) = -1.$

(d) What is the unique solution to the following initial-value problem?

$$y' = -2xy^2, \quad y(0) = 0.$$

16. Consider the initial-value problem:

$$y' = y(y-1), \quad y(x_0) = y_0.$$

(a) Verify that the hypotheses of the existence and uniqueness theorem are satisfied for this initial-value problem for any x_0, y_0. This establishes that the initial-value problem always has a unique solution on some interval containing x_0.

(b) By inspection, determine all equilibrium solutions to the differential equation.

(c) Determine the regions in the xy-plane where the solution curves are concave up, and determine those regions where they are concave down.

(d) Sketch the slope field for the differential equation, and determine all values of y_0 for which the initial-value problem has bounded solutions. On your slope field, sketch representative solution curves in the three cases $y_0 < 0, 0 < y_0 < 1$, and $y_0 > 1$.

For Problems 17–24, sketch the slope field and some representative solution curves for the given differential equation.

17. $y' = 4x$.

18. $y' = 1/x$.

19. $y' = x + y$.

20. $y' = x/y$.

21. $y' = -4x/y$.

22. $y' = x^2 y$.

23. $y' = x^2 \cos y$.

24. $y' = x^2 + y^2$.

25. According to Newton's law of cooling (see Section 1.1), the temperature of an object at time t is governed by the differential equation

$$\frac{dT}{dt} = -k(T - T_m),$$

where T_m is the temperature of the surrounding medium, and k is a constant. Consider the case when $T_m = 70$ and $k = 1/80$. Sketch the corresponding slope field and some representative solution curves. What happens to the temperature of the object as $t \to \infty$? Note that this result is independent of the initial temperature of the object.

For Problems 26–31, determine the slope field and some representative solution curves for the given differential equation.

26. ◊ $y' = -2xy$.

27. ◊ $y' = \dfrac{x \sin x}{1 + y^2}$.

28. ◊ $y' = 3x - y$.

29. ◊ $y' = 2x^2 \sin y$.

30. ◊ $y' = \dfrac{2 + y^2}{3 + 0.5x^2}$.

31. ◊ $y' = \dfrac{1 - y^2}{2 + 0.5x^2}$.

32. ◊

(a) Determine the slope field for the differential equation

$$y' = x^{-1}(3 \sin x - y)$$

on the interval (0, 10].

(b) Plot the solution curves corresponding to each of the following initial conditions:

$$y(0.5) = 0, \quad y(1) = -1,$$

$$y(1) = 2, \quad y(3) = 0.$$

What do you conclude about the behavior as $x \to 0^+$ of solutions to the differential equation?

(c) Plot the solution curve corresponding to the initial condition $y(\pi/2) = 6/\pi$. How does this fit in with your answer to part (b)?

(d) Describe the behavior of the solution curves for large positive x.

33. ◇ Consider the family of curves $y = kx^2$, where k is a constant.

(a) Show that the differential equation of the family of orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{x}{2y}.$$

(b) On the same axes sketch the slope field for the preceding differential equation and several members of the given family of curves. Describe the family of orthogonal trajectories.

34. ◇ Consider the differential equation

$$\frac{di}{dt} + ai = b,$$

where a and b are constants. By drawing the slope fields corresponding to various values of a and b, formulate a conjecture regarding the value of

$$\lim_{t\to\infty} i(t).$$

1.4 Separable Differential Equations

In the previous section we analyzed first-order differential equations using qualitative techniques. We now begin an analytical study of these differential equations by developing some solution techniques that enable us to determine the exact solution to certain types of differential equations. The simplest differential equations for which a solution technique can be obtained are the so-called separable equations, which are defined as follows:

DEFINITION 1.4.1

A first-order differential equation is called **separable** if it can be written in the form

$$p(y)\frac{dy}{dx} = q(x). \tag{1.4.1}$$

The solution technique for a separable differential equation is given in Theorem 1.4.2.

Theorem 1.4.2 If $p(y)$ and $q(x)$ are continuous, then Equation (1.4.1) has the general solution

$$\int p(y)\,dy = \int q(x)\,dx + c, \tag{1.4.2}$$

where c is an arbitrary constant.

Proof We use the chain rule for derivatives to rewrite Equation (1.4.1) in the equivalent form

$$\frac{d}{dx}\left(\int p(y)\,dy\right) = q(x).$$

Integrating both sides of this equation with respect to x yields Equation (1.4.2). ■