(c) Plot the solution curve corresponding to the initial condition $y(\pi/2) = 6/\pi$. How does this fit in with your answer to part (b)?

(d) Describe the behavior of the solution curves for large positive x.

33. ◇ Consider the family of curves $y = kx^2$, where k is a constant.

(a) Show that the differential equation of the family of orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{x}{2y}.$$

(b) On the same axes sketch the slope field for the preceding differential equation and several members of the given family of curves. Describe the family of orthogonal trajectories.

34. ◇ Consider the differential equation

$$\frac{di}{dt} + ai = b,$$

where a and b are constants. By drawing the slope fields corresponding to various values of a and b, formulate a conjecture regarding the value of

$$\lim_{t\to\infty} i(t).$$

1.4 Separable Differential Equations

In the previous section we analyzed first-order differential equations using qualitative techniques. We now begin an analytical study of these differential equations by developing some solution techniques that enable us to determine the exact solution to certain types of differential equations. The simplest differential equations for which a solution technique can be obtained are the so-called separable equations, which are defined as follows:

DEFINITION 1.4.1

A first-order differential equation is called **separable** if it can be written in the form

$$p(y)\frac{dy}{dx} = q(x). \tag{1.4.1}$$

The solution technique for a separable differential equation is given in Theorem 1.4.2.

Theorem 1.4.2 If $p(y)$ and $q(x)$ are continuous, then Equation (1.4.1) has the general solution

$$\int p(y)\,dy = \int q(x)\,dx + c, \tag{1.4.2}$$

where c is an arbitrary constant.

Proof We use the chain rule for derivatives to rewrite Equation (1.4.1) in the equivalent form

$$\frac{d}{dx}\left(\int p(y)\,dy\right) = q(x).$$

Integrating both sides of this equation with respect to x yields Equation (1.4.2). ■

Remark In differential form, Equation (1.4.1) can be written as

$$p(y)\,dy = q(x)\,dx,$$

and the general solution (1.4.2) is obtained by integrating the left-hand side with respect to y and the right-hand side with respect to x. This is the general procedure for solving separable equations.

Example 1.4.3 Solve $(1 + y^2)\dfrac{dy}{dx} = x \cos x$.

Solution: By inspection we see that the differential equation is separable. Integrating both sides of the differential equation yields

$$\int (1 + y^2)\,dy = \int x \cos x\,dx + c.$$

Using integration by parts to evaluate the integral on the right-hand side, we obtain

$$y + \tfrac{1}{3}y^3 = x \sin x + \cos x + c,$$

or equivalently

$$y^3 + 3y = 3(x \sin x + \cos x) + c_1,$$

where $c_1 = 3c$. As often happens with separable differential equations, the solution is given in implicit form. □

In general, the differential equation $dy/dx = f(x)g(y)$ is separable, since it can be written as

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x),$$

which is of the form of Equation (1.4.1) with $p(y) = 1/g(y)$. It is important to note, however, that in writing the given differential equation in this way, we have assumed that $g(y) \neq 0$. Thus the general solution to the resulting differential equation may not include solutions of the original equation corresponding to any values of y for which $g(y) = 0$. (These are the equilibrium solutions for the original differential equation.) We will illustrate with an example.

Example 1.4.4 Find all solutions to

$$y' = -2y^2 x. \tag{1.4.3}$$

Solution: Separating the variables yields

$$y^{-2}dy = -2x\,dx. \tag{1.4.4}$$

Integrating both sides, we obtain

$$-y^{-1} = -x^2 + c$$

so that

$$y(x) = \frac{1}{x^2 - c}. \tag{1.4.5}$$

This is the general solution to Equation (1.4.4). It is not the general solution to Equation (1.4.3), since there is no value of the constant c for which $y(x) = 0$, whereas by inspection we see $y(x) = 0$ is a solution to Equation (1.4.3). This solution is not contained in (1.4.5), since in separating the variables, we divided by y and hence assumed implicitly that $y \neq 0$. Thus the solutions to Equation (1.4.3) are

$$y(x) = \frac{1}{x^2 - c} \qquad \text{and} \qquad y(x) = 0.$$

The slope field for the given differential equation is depicted in Figure 1.4.1, together with some representative solution curves. □

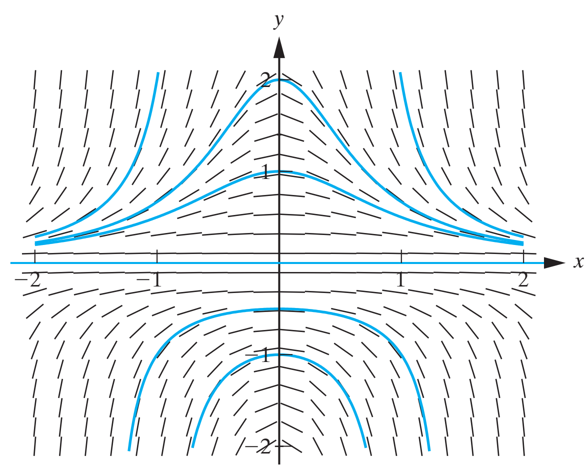

Figure 1.4.1: The slope field and some solution curves for the differential equation $dy/dx = -2xy^2$.

Many difficulties that students encounter with first-order differential equations arise not from the solution techniques themselves, but in the algebraic simplifications that are used to obtain a simple form for the resulting solution. We will explicitly illustrate some of the standard simplifications using the differential equation

$$\frac{dy}{dx} = -2xy.$$

First notice that $y(x) = 0$ is an equilibrium solution to the differential equation. Consequently, no other solution curves can cross the x-axis. For $y \neq 0$ we can separate the variables to obtain

$$\frac{1}{y} dy = -2x\, dx. \tag{1.4.6}$$

Integrating this equation yields

$$\ln |y| = -x^2 + c.$$

Exponentiating both sides of this solution gives

$$|y| = e^{-x^2+c},$$

or equivalently,

$$|y| = e^c e^{-x^2}.$$

We now introduce a new constant c_1 defined by $c_1 = e^c$. Then the preceding expression for $|y|$ reduces to

$$|y| = c_1 e^{-x^2}. \tag{1.4.7}$$

Notice that c_1 is a positive constant. This is a perfectly acceptable form for the solution. However, a redefinition of the integration constant can be used to eliminate the absolute-value bars as follows. According to (1.4.7), the solution to the differential equation is

$$y(x) = \begin{cases} c_1 e^{-x^2}, & \text{if } y > 0, \\ -c_1 e^{-x^2}, & \text{if } y < 0. \end{cases} \tag{1.4.8}$$

We can now define a new constant c_2, by

$$c_2 = \begin{cases} c_1, & \text{if } y > 0, \\ -c_1, & \text{if } y < 0, \end{cases}$$

in terms of which the solutions given in (1.4.8) can be combined into the single formula

$$y(x) = c_2 e^{-x^2}. \tag{1.4.9}$$

The appropriate sign for c_2 will be determined from the initial conditions. For example, the initial condition $y(0) = 1$ would require that $c_2 = 1$, with corresponding unique solution

$$y(x) = e^{-x^2}.$$

Similarly the initial condition $y(0) = -1$ leads to $c_2 = -1$, so that

$$y(x) = -e^{-x^2}.$$

We make one further point about the solution (1.4.9). In obtaining the separable form (1.4.6), we divided the given differential equation by y, and so the derivation of the solution obtained assumes that $y \neq 0$. However, as we have already noted, $y(x) = 0$ is indeed a solution to this differential equation. Formally this solution is the special case $c_2 = 0$ in (1.4.9) and corresponds to the initial condition $y(0) = 0$. Thus (1.4.9) does give the general solution to the differential equation, provided we allow c_2 to assume the value zero. The slope field for the differential equation, together with some particular solution curves, is shown in Figure 1.4.2.

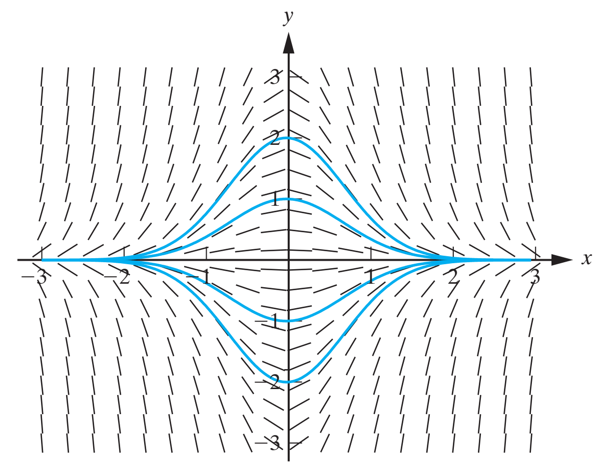

Figure 1.4.2: Slope field and some solution curves for the differential equation $dy/dx = -2xy$.

Example 1.4.5

An object of mass m falls from rest, starting at a point near the earth's surface. Assuming that the air resistance is proportional to the velocity of the object, determine the subsequent motion.

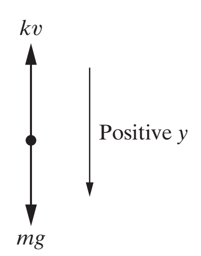

Figure 1.4.3: Particle falling under the influence of gravity and air resistance.

Solution: Let $y(t)$ be the distance traveled by the object at time t from the point it was released, and let the positive y-direction be downward. Then, $y(0) = 0$, and the velocity of the object is $v(t) = dy/dt$. Since the object was dropped from rest, we have $v(0) = 0$. The forces acting on the object are those due to gravity, $F_g = mg$, and the force due to air resistance, $F_r = -kv$, where k is a positive constant (see Figure 1.4.3). According to Newton's second law, the differential equation describing the motion of the object is

$$m\frac{dv}{dt} = F_g + F_r = mg - kv.$$

We are also given the initial condition $v(0) = 0$. Thus the initial-value problem governing the behavior of v is

$$\begin{cases} m\dfrac{dv}{dt} = mg - kv, \\ v(0) = 0. \end{cases} \tag{1.4.10}$$

Separating the variables in Equation (1.4.10) yields

$$\frac{m}{mg - kv}\,dv = dt,$$

which can be integrated directly to obtain

$$-\frac{m}{k}\ln|mg - kv| = t + c.$$

Multiplying both sides of this equation by $-k/m$ and exponentiating the result yields

$$|mg - kv| = c_1 e^{-(k/m)t},$$

where $c_1 = e^{-ck/m}$. By redefining the constant c_1, we can write this in the equivalent form

$$mg - kv = c_2 e^{-(k/m)t}.$$

Hence,

$$v(t) = \frac{mg}{k} - c_3 e^{-(k/m)t}, \tag{1.4.11}$$

where $c_3 = c_2/k$. Imposing the initial condition $v(0) = 0$ yields

$$c_3 = \frac{mg}{k}.$$

So the solution to the initial-value problem (1.4.10) is

$$v(t) = \frac{mg}{k}\left[1 - e^{-(k/m)t}\right]. \tag{1.4.12}$$

Notice that the velocity does not increase indefinitely, but approaches a so-called limiting velocity v_L defined by

$$v_L = \lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{mg}{k}\left[1 - e^{-(k/m)t}\right] = \frac{mg}{k}.$$

The behavior of the velocity as a function of time is shown in Figure 1.4.4. Owing to the negative exponent in (1.4.11), we see that this result is independent of the value of the initial velocity.

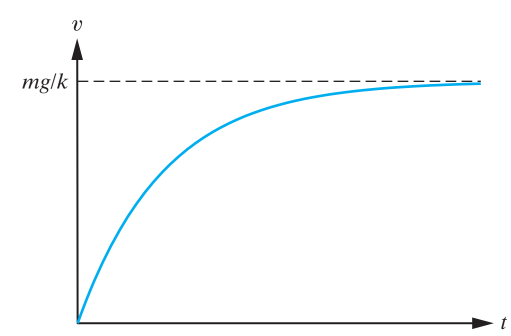

Figure 1.4.4: The behavior of the velocity of the object in Example 1.4.5.

Since $dy/dt = v$, it follows from (1.4.12) that the position of the object at time t can be determined by solving the initial-value problem

$$\frac{dy}{dt} = \frac{mg}{k}\left[1 - e^{-(k/m)t}\right], \qquad y(0) = 0.$$

The differential equation can be integrated directly to obtain

$$y(t) = \frac{mg}{k}\left[t + \frac{m}{k}e^{-(k/m)t}\right] + c.$$

Imposing the initial condition $y(0) = 0$ yields

$$c = -\frac{m^2 g}{k^2},$$

so that

$$y(t) = \frac{mg}{k}\left\{t + \frac{m}{k}\left[e^{-(k/m)t} - 1\right]\right\}.$$

□

Example 1.4.6 A hot metal bar whose temperature is 350°F is placed in a room whose temperature is constant at 70°F. After two minutes, the temperature of the bar is 210°F. Using Newton's law of cooling, determine

1. the temperature of the bar after four minutes.
2. the time required for the bar to cool to 100°F.

Solution: According to Newton's law of cooling (see Section 1.1), the temperature of the object at time t is governed by the differential equation

$$\frac{dT}{dt} = -k(T - T_m), \tag{1.4.13}$$

where, from the statement of the problem,

$$T_m = 70°\text{F}, \qquad T(0) = 350°\text{F}, \qquad T(2) = 210°\text{F}.$$

Substituting for T_m in Equation (1.4.13), we have the separable equation

$$\frac{dT}{dt} = -k(T - 70).$$

Separating the variables yields

$$\frac{1}{T-70}\,dT = -k\,dt,$$

which we can integrate immediately to obtain

$$\ln|T - 70| = -kt + c.$$

Exponentiating both sides and solving for T yields

$$T(t) = 70 + c_1 e^{-kt}, \tag{1.4.14}$$

where we have redefined the integration constant. The two constants c_1 and k can be determined from the given auxiliary conditions as follows. The condition $T(0) = 350°$F requires that $350 = 70 + c_1$. Hence, $c_1 = 280$. Substituting this value for c_1 into (1.4.14) yields

$$T(t) = 70(1 + 4e^{-kt}). \tag{1.4.15}$$

Consequently, $T(2) = 210°$F if and only if

$$210 = 70(1 + 4e^{-2k}),$$

so that $e^{-2k} = \frac{1}{2}$. Hence, $k = \frac{1}{2}\ln 2$, and so, from (1.4.15),

$$T(t) = 70\left[1 + 4e^{-(t/2)\ln 2}\right]. \tag{1.4.16}$$

We can now determine the quantities requested.

1. We have $T(4) = 70(1 + 4e^{-2\ln 2}) = 70\left(1 + 4\cdot\frac{1}{2^2}\right) = 140°\text{F}$.

2. From (1.4.16), $T(t) = 100°\text{F}$ when

$$100 = 70\left[1 + 4e^{-(t/2)\ln 2}\right]$$

—that is, when

$$e^{-(t/2)\ln 2} = \frac{3}{28}.$$

Taking the natural logarithm of both sides and solving for t yields

$$t = \frac{2\ln(28/3)}{\ln 2} \approx 6.4 \text{ minutes.}$$ □

Exercises for 1.4

Skills

- Be able to recognize whether or not a given differential equation is separable.
- Be able to solve separable differential equations.

True-False Review

For Questions 1–9, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. Every differential equation of the form $dy/dx = f(x)g(y)$ is separable.

2. The general solution to a separable differential equation contains one constant whose value can be determined from an initial condition for the differential equation.

3. Newton's law of cooling is a separable differential equation.

4. The differential equation $dy/dx = x^2 + y^2$ is separable.

5. The differential equation $dy/dx = x\sin(xy)$ is separable.

6. The differential equation $\dfrac{dy}{dx} = e^{x+y}$ is separable.

7. The differential equation

$$\frac{dy}{dx} = \frac{1}{x^2(1+y^2)}$$

is separable.

8. The differential equation

$$\frac{dy}{dx} = \frac{x+4y}{4x+y}$$

is separable.

9. The differential equation

$$\frac{dy}{dx} = \frac{x^3y + x^2y^2}{x^2 + xy}$$

is separable.

Problems

For Problems 1–11, solve the given differential equation.

1. $\dfrac{dy}{dx} = 2xy.$

2. $\dfrac{dy}{dx} = \dfrac{y^2}{x^2+1}.$

3. $e^{x+y}dy - dx = 0.$

4. $\dfrac{dy}{dx} = \dfrac{y}{x\ln x}.$

5. $ydx - (x-2)dy = 0.$

6. $\dfrac{dy}{dx} = \dfrac{2x(y-1)}{x^2+3}.$

7. $y - x\dfrac{dy}{dx} = 3 - 2x^2\dfrac{dy}{dx}$.

8. $\dfrac{dy}{dx} = \dfrac{\cos(x-y)}{\sin x \sin y} - 1$.

9. $\dfrac{dy}{dx} = \dfrac{x(y^2-1)}{2(x-2)(x-1)}$.

10. $\dfrac{dy}{dx} = \dfrac{x^2 y - 32}{16 - x^2} + 2$.

11. $(x-a)(x-b)y' - (y-c) = 0$, where a, b, c are constants.

In Problems 12–15, solve the given initial-value problem.

12. $(x^2+1)y' + y^2 = -1$, $\quad y(0) = 1$.

13. $(1-x^2)y' + xy = ax$, $\quad y(0) = 2a$, where a is a constant.

14. $\dfrac{dy}{dx} = 1 - \dfrac{\sin(x+y)}{\sin y \cos x}$, $\quad y(\pi/4) = \pi/4$.

15. $y' = y^3 \sin x$, $\quad y(0) = 0$.

16. One solution to the initial-value problem

$$\frac{dy}{dx} = \frac{2}{3}(y-1)^{1/2}, \quad y(1) = 1$$

is $y(x) = 1$. Determine another solution. Does this contradict the existence and uniqueness theorem (Theorem 1.3.2)? Explain.

17. An object of mass m falls from rest, starting at a point near the earth's surface. Assuming that the air resistance varies as the square of the velocity of the object, a simple application of Newton's second law yields the initial-value problem for the velocity, $v(t)$, of the object at time t:

$$m\frac{dv}{dt} = mg - kv^2, \quad v(0) = 0,$$

where k, m, g are positive constants.

(a) Solve the foregoing initial-value problem for v in terms of t.

(b) Does the velocity of the object increase indefinitely? Justify.

(c) Determine the position of the object at time t.

18. Find the equation of the curve that passes through the point $(0, \frac{1}{2})$ and whose slope at each point (x, y) is $-x/4y$.

19. Find the equation of the curve that passes through the point $(3, 1)$ and whose slope at each point (x, y) is e^{x-y}.

20. Find the equation of the curve that passes through the point $(-1, 1)$ and whose slope at each point (x, y) is x^2y^2.

21. At time t, the velocity $v(t)$ of an object moving in a straight line satisfies

$$\frac{dv}{dt} = -(1+v^2). \tag{1.4.17}$$

(a) Show that

$$\tan^{-1}(v) = \tan^{-1}(v_0) - t,$$

where v_0 denotes the velocity of the object at time $t = 0$ (and we assume $v_0 > 0$). Hence prove that the object comes to rest after a finite time $\tan^{-1}(v_0)$. Does the object remain at rest?

(b) Use the chain rule to show that (1.4.17) can be written as

$$v\frac{dv}{dx} = -(1+v^2),$$

where $x(t)$ denotes the distance traveled by the object at time t, from its position at $t = 0$. Determine the distance traveled by the object when it first comes to rest.

22. The differential equation governing the velocity of an object is

$$\frac{dv}{dt} = -kv^n,$$

where $k > 0$ and n are constants. At $t = 0$, the object is set in motion with velocity v_0.

(a) Show that the object comes to rest in a finite time if and only if $n < 1$, and determine the maximum distance traveled by the object in this case.

(b) If $1 \le n < 2$, show that the maximum distance traveled by the object in a finite time is less than

$$\frac{v_0^{2-n}}{(2-n)k}.$$

(c) If $n \ge 2$, show that there is no limit to the distance that the object can travel.

23. The pressure p, and density, ρ, of the atmosphere at a height y above the earth's surface are related by

$$dp = -g\rho\, dy.$$

Assuming that p and ρ satisfy the adiabatic equation of state $p = p_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma}$, where $\gamma \neq 1$ is a constant and p_0 and ρ_0 denote the pressure and density at the earth's surface, respectively, show that

$$p = p_0 \left[1 - \frac{(\gamma - 1)}{\gamma} \cdot \frac{\rho_0 g y}{p_0}\right]^{\gamma/(\gamma-1)}.$$

24. An object whose temperature is 615°F is placed in a room whose temperature is 75°F. At 4 p.m. the temperature of the object is 135°F, and an hour later its temperature is 95°F. At what time was the object placed in the room?

25. A flammable substance whose initial temperature is 50°F is inadvertently placed in a hot oven whose temperature is 450°F. After 20 minutes, the substance's temperature is 150°F. Find the temperature of the substance after 40 minutes. Assuming that the substance ignites when its temperature reaches 350°F, find the time of combustion.

26. At 2 p.m. on a cool (34°F) afternoon in March, Sherlock Holmes measured the temperature of a dead body to be 38°F. One hour later, the temperature was 36°F. After a quick calculation using Newton's law of cooling, and taking the normal temperature of a living body to be 98°F, Holmes concluded that the time of death was 10 a.m. Was Holmes right?

27. At 4 p.m., a hot coal was pulled out of a furnace and allowed to cool at room temperature (75°F). If, after 10 minutes, the temperature of the coal was 415°F, and after 20 minutes, its temperature was 347°F, find the following:

(a) The temperature of the furnace.

(b) The time when the temperature of the coal was 100°F.

28. A hot object is placed in a room whose temperature is 72°F. After one minute the temperature of the object is 150°F and its rate of change of temperature is 20°F per minute. Find the initial temperature of the object and the rate at which its temperature is changing after 10 minutes.

1.5 Some Simple Population Models

In this section we consider two important models of population growth whose mathematical formulation leads to separable differential equations.

Malthusian Growth

The simplest mathematical model of population growth is obtained by assuming that the rate of increase of the population at any time is proportional to the size of the population at that time. If we let $P(t)$ denote the population at time t, then

$$\frac{dP}{dt} = kP,$$

where k is a positive constant. Separating the variables and integrating yields

$$P(t) = P_0 e^{kt}, \tag{1.5.1}$$

where P_0 denotes the population at $t = 0$. This law predicts an exponential increase in the population with time, which gives a reasonably accurate description of the growth of certain algae, bacteria, and cell cultures. It is called the **Malthusian growth model**. The time taken for such a culture to double in size is called the **doubling time**. This is the time, t_d, when $P(t_d) = 2P_0$. Substituting into (1.5.1) yields

$$2P_0 = P_0 e^{kt_d}.$$

Dividing both sides by P_0 and taking logarithms, we find

$$kt_d = \ln 2,$$