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23. The pressure p , and density, ρ , of the atmosphere at a height y above the earth's surface are related by

$$dp = -g\rho \, dy.$$

Assuming that p and ρ satisfy the adiabatic equation of state $p = p_0 \left(\frac{\rho}{\rho_0} \right)^\gamma$, where $\gamma \neq 1$ is a constant and p_0 and ρ_0 denote the pressure and density at the earth's surface, respectively, show that

$$p = p_0 \left[1 - \frac{(\gamma - 1)}{\gamma} \cdot \frac{\rho_0 g y}{p_0} \right]^{\gamma/(\gamma-1)}.$$

24. An object whose temperature is 615°F is placed in a room whose temperature is 75°F . At 4 p.m. the temperature of the object is 135°F , and an hour later its temperature is 95°F . At what time was the object placed in the room?
25. A flammable substance whose initial temperature is 50°F is inadvertently placed in a hot oven whose temperature is 450°F . After 20 minutes, the substance's temperature is 150°F . Find the temperature of the substance after 40 minutes. Assuming that the substance ignites when its temperature reaches 350°F , find the time of combustion.

26. At 2 p.m. on a cool (34°F) afternoon in March, Sherlock Holmes measured the temperature of a dead body to be 38°F . One hour later, the temperature was 36°F . After a quick calculation using Newton's law of cooling, and taking the normal temperature of a living body to be 98°F , Holmes concluded that the time of death was 10 a.m. Was Holmes right?

27. At 4 p.m., a hot coal was pulled out of a furnace and allowed to cool at room temperature (75°F). If, after 10 minutes, the temperature of the coal was 415°F , and after 20 minutes, its temperature was 347°F , find the following:

- (a) The temperature of the furnace.
(b) The time when the temperature of the coal was 100°F .

28. A hot object is placed in a room whose temperature is 72°F . After one minute the temperature of the object is 150°F and its rate of change of temperature is 20°F per minute. Find the initial temperature of the object and the rate at which its temperature is changing after 10 minutes.

1.5 Some Simple Population Models

In this section we consider two important models of population growth whose mathematical formulation leads to separable differential equations.

Malthusian Growth

The simplest mathematical model of population growth is obtained by assuming that the rate of increase of the population at any time is proportional to the size of the population at that time. If we let $P(t)$ denote the population at time t , then

$$\frac{dP}{dt} = kP,$$

where k is a positive constant. Separating the variables and integrating yields

$$P(t) = P_0 e^{kt}, \quad (1.5.1)$$

where P_0 denotes the population at $t = 0$. This law predicts an exponential increase in the population with time, which gives a reasonably accurate description of the growth of certain algae, bacteria, and cell cultures. It is called the **Malthusian growth model**. The time taken for such a culture to double in size is called the **doubling time**. This is the time, t_d , when $P(t_d) = 2P_0$. Substituting into (1.5.1) yields

$$2P_0 = P_0 e^{kt_d}.$$

Dividing both sides by P_0 and taking logarithms, we find

$$kt_d = \ln 2,$$

so that the doubling time is

$$t_d = \frac{1}{k} \ln 2.$$

Example 1.5.1

The number of bacteria in a certain culture grows at a rate that is proportional to the number present. If the number increased from 500 to 2000 in 2 hours, determine

1. the number present after 12 hours.
2. the doubling time.

Solution: The behavior of the system is governed by the differential equation

$$\frac{dP}{dt} = kP,$$

so that

$$P(t) = P_0 e^{kt},$$

where the time t is measured in hours. Taking $t = 0$ as the time when the population was 500, we have $P_0 = 500$. Thus,

$$P(t) = 500e^{kt}.$$

Further, $P(2) = 2000$ implies that

$$2000 = 500e^{2k},$$

so that

$$k = \frac{1}{2} \ln 4 = \ln 2.$$

Consequently,

$$P(t) = 500e^{t \ln 2}.$$

1. The number of bacteria present after 12 hours is therefore

$$P(12) = 500e^{12 \ln 2} = 500(2^{12}) = 2,048,000.$$

2. The doubling time of the system is

$$t_d = \frac{1}{k} \ln 2 = 1 \text{ hour.} \quad \square$$

Logistic Population Model

The Malthusian growth law (1.5.1) does not provide an accurate model for the growth of a population over a long time period. To obtain a more realistic model we need to take account of the fact that as the population increases, several factors will begin to affect the growth rate. For example, there will be increased competition for the limited resources that are available, increases in disease, and overcrowding of the limited available space, all of which would serve to slow the growth rate. In order to model this situation mathematically, we modify the differential equation leading to the simple exponential growth

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law by adding in a term that slows the growth down as the population increases. If we consider a closed environment (neglecting factors such as immigration and emigration), then the rate of change of population can be modeled by the differential equation

$$\frac{dP}{dt} = [B(t) - D(t)]P,$$

where $B(t)$ and $D(t)$ denote the birth rate and death rate per individual, respectively. The simple exponential law corresponds to the case when $B(t) = k$ and $D(t) = 0$. In the more general situation of interest now, the increased competition as the population grows will result in a corresponding increase in the death rate per individual. Perhaps the simplest way to take account of this is to assume that the death rate per individual is directly proportional to the instantaneous population, and that the birth rate per individual remains constant. The resulting initial-value problem governing the population growth can then be written as

$$\frac{dP}{dt} = (B_0 - D_0 P)P, \quad P(0) = P_0,$$

where B_0 and D_0 are *positive* constants. It is useful to write the differential equation in the equivalent form

$$\frac{dP}{dt} = r \left(1 - \frac{P}{C} \right) P, \quad (1.5.2)$$

where $r = B_0$, and $C = B_0/D_0$. Equation (1.5.2) is called the **logistic equation**, and the corresponding population model is called the **logistic model**. The differential equation (1.5.2) is separable and can be solved without difficulty. Before doing that, however, we give a qualitative analysis of the differential equation.

The constant C in Equation (1.5.2) is called the **carrying capacity** of the population. We see from Equation (1.5.2) that if $P < C$, then $dP/dt > 0$ and the population increases, whereas if $P > C$, then $dP/dt < 0$ and the population decreases. We can therefore interpret C as representing the maximum population that the environment can sustain. We note that $P(t) = C$ is an equilibrium solution to the differential equation, as is $P(t) = 0$. The isoclines for Equation (1.5.2) are determined from

$$r \left(1 - \frac{P}{C} \right) P = k,$$

where k is a constant. This can be written as

$$P^2 - CP + \frac{kC}{r} = 0,$$

so that the isoclines are the lines

$$P = \frac{1}{2} \left(C \pm \sqrt{C^2 - \frac{4kC}{r}} \right).$$

This tells us that the slopes of the solution curves satisfy

$$C^2 - \frac{4kC}{r} \geq 0,$$

so that

$$k \leq rC/4.$$

Furthermore, the largest value that the slope can assume is $k = rC/4$, which corresponds to $P = C/2$. We also note that the slope approaches zero as the solution curves approach the equilibrium solutions $P(t) = 0$ and $P(t) = C$. Differentiating Equation (1.5.2) yields

$$\frac{d^2 P}{dt^2} = r \left[\left(1 - \frac{P}{C}\right) \frac{dP}{dt} - \frac{P}{C} \frac{dP}{dt} \right] = r \left(1 - 2\frac{P}{C}\right) \frac{dP}{dt} = \frac{r^2}{C^2} (C - 2P)(C - P)P,$$

where we have substituted for dP/dt from (1.5.2) and simplified the result. Since $P = C$ and $P = 0$ are solutions to the differential equation (1.5.2), the only points of inflection occur along the line $P = C/2$. The behavior of the concavity is therefore given by the following schematic:

sign of P'':	+ + + + - - - - + + + +
P-interval:	0 $C/2$ C

This information determines the general behavior of the solution curves to the differential equation (1.5.2). Figure 1.5.1 gives a Maple plot of the slope field and some representative solution curves. Of course, such a figure could have been constructed by hand, using the information we have obtained. From Figure 1.5.1, we see that if the initial population is less than the carrying capacity, then the population increases monotonically toward the carrying capacity. Similarly, if the initial population is bigger than the carrying capacity, then the population monotonically decreases toward the carrying capacity. Once more this illustrates the power of the qualitative techniques that have been introduced for analyzing first-order differential equations.

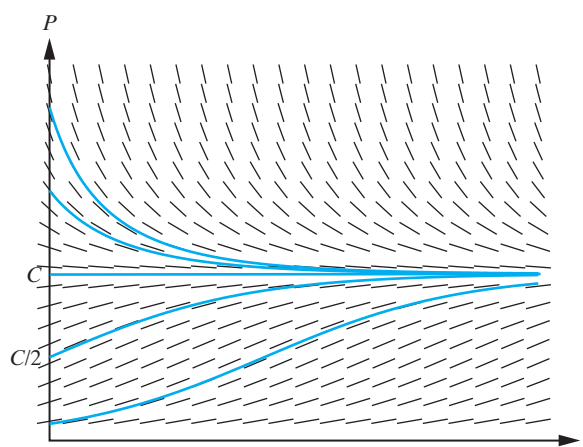


Figure 1.5.1: Representative slope field and some approximate solution curves for the logistic equation.

We turn now to obtaining an analytical solution to the differential equation (1.5.2). Separating the variables in Equation (1.5.2) and integrating yields

$$\int \frac{C}{P(C - P)} dP = rt + c_1,$$

where c_1 is an integration constant. Using a partial-fraction decomposition on the left-hand side, we find

$$\int \left(\frac{1}{P} + \frac{1}{C - P} \right) dP = rt + c_1,$$

which upon integration gives

$$\ln \left| \frac{P}{C - P} \right| = rt + c_1.$$

Exponentiating, and redefining the integration constant, yields

$$\frac{P}{C - P} = c_2 e^{rt},$$

which can be solved algebraically for P to obtain

$$P(t) = \frac{c_2 C e^{rt}}{1 + c_2 e^{rt}},$$

or equivalently,

$$P(t) = \frac{c_2 C}{c_2 + e^{-rt}}.$$

Imposing the initial condition $P(0) = P_0$, we find that $c_2 = P_0/(C - P_0)$. Inserting this value of c_2 into the preceding expression for $P(t)$ yields

$$P(t) = \frac{C P_0}{P_0 + (C - P_0)e^{-rt}}. \quad (1.5.3)$$

We make two comments regarding this formula. First, we see that, owing to the negative exponent of the exponential term in the denominator, as $t \rightarrow \infty$ the population does indeed tend to the carrying capacity C independently of the initial population P_0 . Second, by writing (1.5.3) in the equivalent form

$$P(t) = \frac{P_0}{P_0/C + (1 - P_0/C)e^{-rt}},$$

it follows that if P_0 is very small compared to the carrying capacity, then for small t the terms involving P_0 in the denominator can be neglected, leading to the approximation

$$P(t) \approx P_0 e^{rt}.$$

Consequently, in this case, the Malthusian population model does approximate the logistic model for small time intervals.

Although we now have a formula for the solution to the logistic population model, the qualitative analysis is certainly enlightening with regard to the general overall properties of the solution. Of course if we want to investigate specific details of a particular model, then we use the corresponding exact solution (1.5.3).

Example 1.5.2

The initial population (measured in thousands) of a city is 20. After 10 years this has increased to 50.87, and after 15 years to 78.68. Use the logistic model to predict the population after 30 years.

Solution: In this problem we have $P_0 = P(0) = 20$, $P(10) = 50.87$, $P(15) = 78.68$, and we wish to find $P(30)$. Substituting for P_0 into Equation (1.5.3) yields

$$P(t) = \frac{20C}{20 + (C - 20)e^{-rt}}. \quad (1.5.4)$$

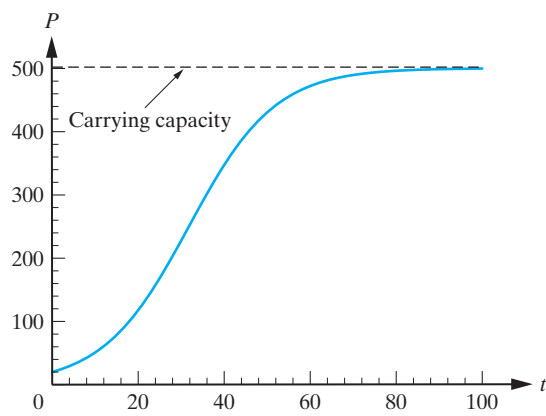


Figure 1.5.2: Solution curve corresponding to the population model in Example 1.5.2. The population is measured in thousands of people.

Imposing the two remaining auxiliary conditions leads to the following pair of equations for determining r and C :

$$\begin{aligned} 50.87 &= \frac{20C}{20 + (C - 20)e^{-10r}}, \\ 78.68 &= \frac{20C}{20 + (C - 20)e^{-15r}}. \end{aligned}$$

This is a pair of nonlinear equations that are tedious to solve by hand. We therefore turn to technology. Using the algebraic capabilities of Maple, we find that

$$r \approx 0.1, \quad C \approx 500.37.$$

Substituting these values of r and C in Equation (1.5.4) yields

$$P(t) = \frac{10007.4}{20 + 480.37e^{-0.1t}}.$$

Accordingly, the predicted value of the population after 30 years is

$$P(30) = \frac{10007.4}{20 + 480.37e^{-3}} \approx 227.87.$$

A sketch of $P(t)$ is given in Figure 1.5.2. □

Exercises for 1.5

Key Terms

Malthusian growth model, Doubling time, Logistic growth model, Carrying capacity.

Skills

- Be able to solve the basic differential equations describing the Malthusian and logistic population growth models.
- Be able to solve word problems involving initial conditions, doubling time, etc., for the Malthusian and logistic population growth models.
- Be able to compute the carrying capacity for a logistic population model.
- Be able to discuss the qualitative behavior of a population governed by a Malthusian or logistic model, based on initial values, doubling time, and so on as a function of time.

True-False Review

For Questions 1–10, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. A population whose growth rate at any given time is proportional to its size at that time obeys the Malthusian growth model.
2. If a population obeys the logistic growth model, then its size can never exceed the carrying capacity of the population.
3. The differential equations which describe population growth according to the Malthusian model and the logistic model are both separable.
4. The rate of change of a population whose growth is described with the logistic model eventually tends toward zero, regardless of the initial population.
5. If the doubling time of a population governed by the Malthusian growth model is five minutes, then the initial population increases 64-fold in a half-hour.
6. If a population whose growth is based on the Malthusian growth model has a doubling time of 10 years, then it takes approximately 30–40 years in order for the initial population size to increase tenfold.
7. The population growth rate according to the Malthusian growth model is always constant.
8. The logistic population model always has exactly two equilibrium solutions.
9. The concavity of the graph of population governed by the logistic model changes if and only if the initial population is less than the carrying capacity.
10. The concavity of the graph of a population governed by the Malthusian growth model never changes, regardless of the initial population.

Problems

1. The number of bacteria in a culture grows at a rate proportional to the number present. Initially there were 10 bacteria in the culture. If the doubling time of the culture is 3 hours, find the number of bacteria that were present after 24 hours.
2. The number of bacteria in a culture grows at a rate proportional to the number present. After 10 hours, there were 5000 bacteria present, and after 12 hours, 6000

bacteria. Determine the initial size of the culture and the doubling time of the population.

3. A certain cell culture has a doubling time of 4 hours. Initially there were 2000 cells present. Assuming an exponential growth law, determine the time it takes for the culture to contain 10^6 cells.
4. At time t , the population $P(t)$ of a certain city is increasing at a rate proportional to the number of residents in the city at that time. In January 1990 the population of the city was 10,000, and by 1995 it had risen to 20,000.
 - (a) What will the population of the city be at the beginning of the year 2010?
 - (b) In what year will the population reach one million?

In the logistic population model (1.5.3), if $P(t_1) = P_1$ and $P(2t_1) = P_2$, then it can be shown (through some algebra performed tediously by hand, or easily on a computer algebra system) that

$$r = \frac{1}{t_1} \ln \left[\frac{P_2(P_1 - P_0)}{P_0(P_2 - P_1)} \right], \quad (1.5.5)$$

$$C = \frac{P_1[P_1(P_0 + P_2) - 2P_0P_2]}{P_1^2 - P_0P_2}. \quad (1.5.6)$$

These formulas will be used in Problems 5–7.

5. The initial population in a small village is 500. After 5 years this has grown to 800 and after 10 years to 1000. Using the logistic population model, determine the population after 15 years.
6. An animal sanctuary had an initial population of 50 animals. After two years the population was 62 and after four years 76. Using the logistic population model, determine the carrying capacity and the number of animals in the sanctuary after 20 years.
7. (a) Using Equations (1.5.5) and (1.5.6), and the fact that r and C are positive, derive two inequalities that P_0 , P_1 , P_2 must satisfy in order for there to be a solution to the logistic equation satisfying the conditions

$$P(0) = P_0, \quad P(t_1) = P_1, \quad P(2t_1) = P_2.$$
 (b) The initial population in a town is 10,000. After 5 years this has grown to 12,000, and after 10 years to 18,000. Is there a solution to the logistic equation that fits this data?

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8. Of the 1500 passengers, crew, and staff that board a cruise ship, 5 have the flu. After one day of sailing, the number of infected people has risen to 10. Assuming that the rate at which the flu virus spreads is proportional to the product of the number of infected individuals and the number not yet infected, determine how many people will have the flu at the end of the 14-day cruise. Would you like to be a member of the customer relations department for the cruise line the day after the ship docks?

9. Consider the population model

$$\frac{dP}{dt} = r(P - T)P, \quad P(0) = P_0, \quad (1.5.7)$$

where r , T , and P_0 are positive constants.

- Perform a qualitative analysis of the differential equation in the initial-value problem (1.5.7), following the steps used in the text for the logistic equation. Identify the equilibrium solutions, the isoclines, and the behavior of the slope and concavity of the solution curves.
 - Using the information obtained in (a), sketch the slope field for the differential equation and include representative solution curves.
 - What predictions can you make regarding the behavior of the population? Consider the cases $P_0 < T$ and $P_0 > T$. The constant T is called the **threshold level**. Based on your predictions, why is this an appropriate term to use for T ?
10. In the preceding problem, a qualitative analysis of the differential equation in (1.5.7) was carried out. In this problem, we determine the exact solution to the differential equation and verify the predictions from the qualitative analysis.
- Solve the initial-value problem (1.5.7).
 - Using your solution from (a), verify that if $P_0 < T$, then $\lim_{t \rightarrow \infty} P(t) = 0$. What does this mean for the population?
 - Using your solution from (a), verify that if $P_0 > T$, then each solution curve has a vertical asymptote at $t = t_e$, where

$$t_e = \frac{1}{rT} \ln \left(\frac{P_0}{P_0 - T} \right).$$

How do you interpret this result in terms of population growth? Note that this was not obvious from the qualitative analysis performed in the previous problem.

11. As a modification to the population model considered in the previous two problems, suppose that $P(t)$ satisfies the initial-value problem

$$\frac{dP}{dt} = r(C - P)(P - T)P, \quad P(0) = P_0,$$

where r , C , T , P_0 are positive constants, and $0 < T < C$. Perform a qualitative analysis of this model. Sketch the slope field and some representative solution curves in the three cases $0 < P_0 < T$, $T < P_0 < C$, and $P_0 > C$. Describe the behavior of the corresponding solutions.

The next two problems consider the **Gompertz** population model, which is governed by the initial-value problem

$$\frac{dP}{dt} = rP(\ln C - \ln P), \quad P(0) = P_0, \quad (1.5.8)$$

where r , C , and P_0 are positive constants.

- Determine all equilibrium solutions for the differential equation in (1.5.8), and the behavior of the slope and concavity of the solution curves. Use this information to sketch the slope field and some representative solution curves.
- Solve the initial-value problem (1.5.8) and verify that all solutions satisfy $\lim_{t \rightarrow \infty} P(t) = C$.

Problems 14–16 consider the phenomenon of exponential decay. This occurs when a population $P(t)$ is governed by the differential equation

$$\frac{dP}{dt} = kP,$$

where k is a *negative* constant.

14. A population of swans in a wildlife sanctuary is declining due to the presence of dangerous chemicals in the water. If the population of swans is experiencing exponential decay, and if there were 400 swans in the park at the beginning of the summer and 340 swans 30 days later,
- How many swans are in the park 60 days after the start of summer? 100 days after the start of summer?
 - How long does it take for the population of swans to be cut in half? (This is known as the **half-life** of the population.)

1.6 First-Order Linear Differential Equations 49

15. At the conclusion of the Super Bowl, the number of fans remaining in the stadium decreases at a rate proportional to the number of fans in the stadium. Assume that there are 100,000 fans in the stadium at the end of the Super Bowl and ten minutes later there are 80,000 fans in the stadium.
- Thirty minutes after the Super Bowl will there be more or less than 40,000 fans? How do you know this without doing any calculations?
 - What is the half-life (see the previous problem) for the fan population in the stadium?
 - When will there be only 15,000 fans left in the stadium?
 - Explain why the exponential decay model for the population of fans in the stadium is not realistic from a qualitative perspective.
16. Cobalt-60, an isotope used in cancer therapy, decays exponentially with a half-life of 5.2 years (i.e., half the original sample remains after 5.2 years). How long does it take for a sample of cobalt-60 to disintegrate to the extent that only 4% of the original amount remains?
17. $\diamond$ Use some form of technology to solve the pair of equations

$$P_1 = \frac{C P_0}{P_0 + (C - P_0)e^{-rt_1}},$$

$$P_2 = \frac{C P_0}{P_0 + (C - P_0)e^{-2rt_1}},$$

for r and C , and thereby derive the expressions given in Equations (1.5.5) and (1.5.6).

18. $\diamond$ According to data from the U.S. Bureau of the Census, the population (measured in millions of people) of the United States in 1950, 1960, and 1970 was, respectively, 151.3, 179.4, and 203.3.
- Using the 1950 and 1960 population figures, solve the corresponding Malthusian population model.
 - Determine the logistic model corresponding to the given data.
 - On the same set of axes, plot the solution curves obtained in (a) and (b). From your plots, determine the values the different models would have predicted for the population in 1980 and 1990, and compare these predictions to the actual values of 226.54 and 248.71, respectively.
19. $\diamond$ In a period of five years, the population of a city doubles from its initial size of 50 (measured in thousands of people). After ten more years, the population has reached 250. Determine the logistic model corresponding to this data. Sketch the solution curve and use your plot to estimate the time it will take for the population to reach 95% of the carrying capacity.

1.6 First-Order Linear Differential Equations

In this section we derive a technique for determining the general solution to any first-order linear differential equation. This is the most important technique in the chapter.

DEFINITION 1.6.1

A differential equation that can be written in the form

$$a(x)\frac{dy}{dx} + b(x)y = r(x) \quad (1.6.1)$$

where $a(x)$, $b(x)$, and $r(x)$ are functions defined on an interval (a, b) , is called a **first-order linear differential equation**.

We assume that $a(x) \neq 0$ on (a, b) and divide both sides of (1.6.1) by $a(x)$ to obtain the **standard form**

$$\frac{dy}{dx} + p(x)y = q(x), \quad (1.6.2)$$

where $p(x) = b(x)/a(x)$ and $q(x) = r(x)/a(x)$. The idea behind the solution technique