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15. At the conclusion of the Super Bowl, the number of fans remaining in the stadium decreases at a rate proportional to the number of fans in the stadium. Assume that there are 100,000 fans in the stadium at the end of the Super Bowl and ten minutes later there are 80,000 fans in the stadium.
- Thirty minutes after the Super Bowl will there be more or less than 40,000 fans? How do you know this without doing any calculations?
 - What is the half-life (see the previous problem) for the fan population in the stadium?
 - When will there be only 15,000 fans left in the stadium?
 - Explain why the exponential decay model for the population of fans in the stadium is not realistic from a qualitative perspective.
16. Cobalt-60, an isotope used in cancer therapy, decays exponentially with a half-life of 5.2 years (i.e., half the original sample remains after 5.2 years). How long does it take for a sample of cobalt-60 to disintegrate to the extent that only 4% of the original amount remains?
17. \diamond Use some form of technology to solve the pair of equations

$$P_1 = \frac{C P_0}{P_0 + (C - P_0)e^{-rt_1}},$$

$$P_2 = \frac{C P_0}{P_0 + (C - P_0)e^{-2rt_1}},$$

for r and C , and thereby derive the expressions given in Equations (1.5.5) and (1.5.6).

18. \diamond According to data from the U.S. Bureau of the Census, the population (measured in millions of people) of the United States in 1950, 1960, and 1970 was, respectively, 151.3, 179.4, and 203.3.
- Using the 1950 and 1960 population figures, solve the corresponding Malthusian population model.
 - Determine the logistic model corresponding to the given data.
 - On the same set of axes, plot the solution curves obtained in (a) and (b). From your plots, determine the values the different models would have predicted for the population in 1980 and 1990, and compare these predictions to the actual values of 226.54 and 248.71, respectively.
19. \diamond In a period of five years, the population of a city doubles from its initial size of 50 (measured in thousands of people). After ten more years, the population has reached 250. Determine the logistic model corresponding to this data. Sketch the solution curve and use your plot to estimate the time it will take for the population to reach 95% of the carrying capacity.

1.6 First-Order Linear Differential Equations

In this section we derive a technique for determining the general solution to any first-order linear differential equation. This is the most important technique in the chapter.

DEFINITION 1.6.1

A differential equation that can be written in the form

$$a(x) \frac{dy}{dx} + b(x)y = r(x) \quad (1.6.1)$$

where $a(x)$, $b(x)$, and $r(x)$ are functions defined on an interval (a, b) , is called a **first-order linear differential equation**.

We assume that $a(x) \neq 0$ on (a, b) and divide both sides of (1.6.1) by $a(x)$ to obtain the **standard form**

$$\frac{dy}{dx} + p(x)y = q(x), \quad (1.6.2)$$

where $p(x) = b(x)/a(x)$ and $q(x) = r(x)/a(x)$. The idea behind the solution technique

for (1.6.2) is to rewrite the differential equation in the form

$$\frac{d}{dx}[g(x, y)] = F(x)$$

for an appropriate function $g(x, y)$. The general solution to the differential equation can then be obtained by an integration with respect to x . First consider an example.

Example 1.6.2

Solve the differential equation

$$\frac{dy}{dx} + \frac{1}{x}y = e^x, \quad x > 0. \quad (1.6.3)$$

Solution: If we multiply (1.6.3) by x , we obtain

$$x \frac{dy}{dx} + y = xe^x.$$

But, from the product rule for differentiation, the left-hand side of this equation is just the expanded form of $\frac{d}{dx}(xy)$. Thus (1.6.3) can be written in the equivalent form

$$\frac{d}{dx}(xy) = xe^x.$$

Integrating both sides of this equation with respect to x , we obtain

$$xy = xe^x - e^x + c.$$

Dividing by x yields the general solution to (1.6.3) as

$$y(x) = x^{-1}[e^x(x - 1) + c],$$

where c is an arbitrary constant. □

In the preceding example we multiplied the given differential equation by the function $I(x) = x$. This had the effect of reducing the left-hand side of the resulting differential equation to the integrable form

$$\frac{d}{dx}(xy).$$

Motivated by this example, we now consider the possibility of multiplying the general linear differential equation

$$\frac{dy}{dx} + p(x)y = q(x) \quad (1.6.4)$$

by a nonzero function $I(x)$, chosen in such a way that the left-hand side of the resulting differential equation is

$$\frac{d}{dx}[I(x)y].$$

Henceforth we will assume that the functions p and q are continuous on (a, b) . Multiplying the differential equation (1.6.4) by $I(x)$ yields

$$I \frac{dy}{dx} + p(x)Iy = Iq(x). \quad (1.6.5)$$

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Furthermore, from the product rule for derivatives, we know that

$$\frac{d}{dx}(Iy) = I \frac{dy}{dx} + \frac{dI}{dx}y. \quad (1.6.6)$$

Comparing Equations (1.6.5) and (1.6.6), we see that Equation (1.6.5) can indeed be written in the integrable form

$$\frac{d}{dx}(Iy) = Iq(x),$$

provided the function $I(x)$ is a solution to⁵

$$I \frac{dy}{dx} + p(x)Iy = I \frac{dy}{dx} + \frac{dI}{dx}y.$$

This will hold whenever $I(x)$ satisfies the separable differential equation

$$\frac{dI}{dx} = p(x)I. \quad (1.6.7)$$

Separating the variables and integrating yields

$$\ln |I| = \int p(x) dx + c,$$

so that

$$I(x) = c_1 e^{\int p(x) dx},$$

where c_1 is an arbitrary constant. Since we require only one solution to Equation (1.6.7), we set $c_1 = 1$, in which case

$$I(x) = e^{\int p(x) dx}.$$

We can therefore draw the following conclusion.

Multiplying the linear differential equation

$$\frac{dy}{dx} + p(x)y = q(x) \quad (1.6.8)$$

by $I(x) = e^{\int p(x) dx}$ reduces it to the integrable form

$$\frac{d}{dx} \left[e^{\int p(x) dx} y \right] = q(x) e^{\int p(x) dx}. \quad (1.6.9)$$

The general solution to (1.6.8) can now be obtained from (1.6.9) by integration. Formally we have

$$y(x) = e^{-\int p(x) dx} \left[\int q(x) e^{\int p(x) dx} dx + c \right]. \quad (1.6.10)$$

⁵This is obtained by equating the left-hand side of Equation (1.6.5) to the right-hand side of Equation (1.6.6).

Remarks

1. The function $I(x) = e^{\int p(x)dx}$ is called an **integrating factor** for the differential equation (1.6.8), since it enables us to reduce the differential equation to a form that is directly integrable.
2. It is not necessary to memorize (1.6.10). In a specific problem, we first evaluate the integrating factor $e^{\int p(x)dx}$ and then use (1.6.9).

Example 1.6.3

Solve the initial-value problem

$$\frac{dy}{dx} + xy = xe^{x^2/2}, \quad y(0) = 1.$$

Solution: An appropriate integrating factor in this case is

$$I(x) = e^{\int x \, dx} = e^{x^2/2}.$$

Multiplying the given differential equation by I and using (1.6.9) yields

$$\frac{d}{dx}(e^{x^2/2}y) = xe^{x^2}.$$

Integrating both sides with respect to x , we obtain

$$e^{x^2/2}y = \frac{1}{2}e^{x^2} + c.$$

Hence,

$$y(x) = e^{-x^2/2}(\frac{1}{2}e^{x^2} + c).$$

Imposing the initial condition $y(0) = 1$ yields

$$1 = \frac{1}{2} + c,$$

so that $c = \frac{1}{2}$. Thus the required particular solution is

$$y(x) = \frac{1}{2}e^{-x^2/2}(e^{x^2} + 1) = \frac{1}{2}(e^{x^2/2} + e^{-x^2/2}) = \cosh(x^2/2). \quad \square$$

Example 1.6.4

Solve $x \frac{dy}{dx} + 2y = \cos x$, $x > 0$.

Solution: We first write the given differential equation in standard form. Dividing by x yields

$$\frac{dy}{dx} + 2x^{-1}y = x^{-1} \cos x. \tag{1.6.11}$$

An integrating factor is

$$I(x) = e^{\int 2x^{-1}dx} = e^{2 \ln x} = x^2,$$

so that upon multiplying Equation (1.6.11) by I , we obtain

$$\frac{d}{dx}(x^2 y) = x \cos x.$$

Integrating and rearranging gives

$$y(x) = x^{-2}(x \sin x + \cos x + c),$$

where we have used integration by parts on the right-hand side. \square

Example 1.6.5

Solve the initial-value problem

$$y' - y = f(x), \quad y(0) = 0,$$

$$\text{where } f(x) = \begin{cases} 1, & \text{if } x < 1, \\ 2 - x, & \text{if } x \geq 1. \end{cases}$$

Solution: We have sketched $f(x)$ in Figure 1.6.1. An integrating factor for the differential equation is $I(x) = e^{-x}$.

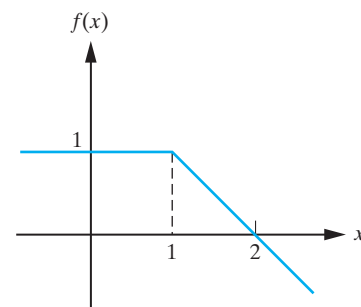


Figure 1.6.1: A sketch of the function $f(x)$ from Example 1.6.5.

Upon multiplication by the integrating factor, the differential equation reduces to

$$\frac{d}{dx}(e^{-x} y) = e^{-x} f(x).$$

We now integrate this differential equation over the interval $[0, x]$. To do so we need to use a dummy integration variable, which we denote by w . We therefore obtain

$$[e^{-w} y(w)]_0^x = \int_0^x e^{-w} f(w) dw,$$

or equivalently,

$$e^{-x} y(x) - y(0) = \int_0^x e^{-w} f(w) dw.$$

Multiplying by e^x and substituting for $y(0) = 0$ yields

$$y(x) = e^x \int_0^x e^{-w} f(w) dw. \quad (1.6.12)$$

Owing to the form of $f(x)$, the value of the integral on the right-hand side will depend on whether $x < 1$ or $x \geq 1$. If $x < 1$, then $f(w) = 1$, and so (1.6.12) can be written as

$$y(x) = e^x \int_0^x e^{-w} dw = e^x(1 - e^{-x}),$$

so that

$$y(x) = e^x - 1, \quad x < 1.$$

If $x \geq 1$, then the interval of integration $[0, x]$ must be split into two parts. From (1.6.12) we have

$$y(x) = e^x \left[\int_0^1 e^{-w} dw + \int_1^x (2 - w)e^{-w} dw \right].$$

A straightforward integration leads to

$$y(x) = e^x \left\{ (1 - e^{-1}) + \left[-2e^{-w} + we^{-w} + e^{-w} \right]_1^x \right\},$$

which simplifies to

$$y(x) = e^x(1 - e^{-1}) + x - 1.$$

The solution to the initial-value problem can therefore be written as

$$y(x) = \begin{cases} e^x - 1, & \text{if } x < 1, \\ e^x(1 - e^{-1}) + x - 1, & \text{if } x \geq 1. \end{cases}$$

A sketch of the corresponding solution curve is given in Figure 1.6.2.

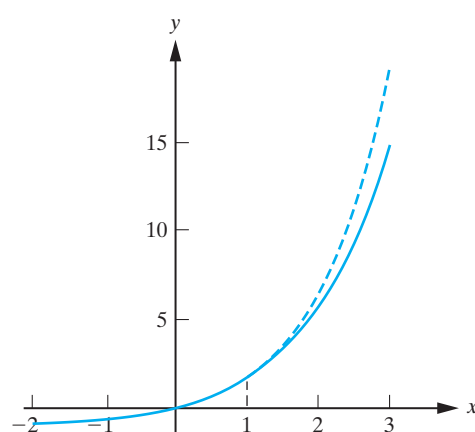


Figure 1.6.2: The solution curve for the initial-value problem in Example 1.6.5. The dashed curve is the continuation of $y(x) = e^x - 1$ for $x > 1$.

Differentiating both branches of this function, we find

$$y'(x) = \begin{cases} e^x, & \text{if } x < 1, \\ e^x(1 - e^{-1}) + 1, & \text{if } x \geq 1. \end{cases} \quad y''(x) = \begin{cases} e^x, & \text{if } x < 1, \\ e^x(1 - e^{-1}), & \text{if } x \geq 1. \end{cases}$$

We see that even though the function f in the original differential equation was not differentiable at $x = 1$, the solution to the initial-value problem has a continuous derivative at that point. The discontinuity in the derivative of the driving term does show up in the second derivative of the solution, as indeed it must. \square

Exercises for 1.6

Key Terms

First-order linear differential equation, Integrating factor.

Skills

- Be able to recognize a first-order linear differential equation.
- Be able to find an integrating factor for a given first-order linear differential equation.
- Be able to solve a first-order linear differential equation.

True-False Review

For Questions 1–5, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. There is a unique integrating factor for a differential equation of the form $y' + p(x)y = q(x)$.
2. An integrating factor for the differential equation $y' + p(x)y = q(x)$ is $e^{\int p(x)dx}$.
3. Upon multiplying the differential equation $y' + p(x)y = q(x)$ by an integrating factor $I(x)$, the differential equation becomes $(I(x) \cdot y)' = q(x)I$.
4. An integrating factor for the differential equation

$$\frac{dy}{dx} = x^2y + \sin x$$

$$\text{is } I(x) = e^{\int x^2 dx}.$$

5. An integrating factor for the differential equation

$$\frac{dy}{dx} = x - \frac{y}{x}$$

$$\text{is } I(x) = 5x.$$

Problems

For Problems 1–14, solve the given differential equation.

1. $\frac{dy}{dx} - y = e^{2x}$.
2. $x^2y' - 4xy = x^7 \sin x, \quad x > 0$.
3. $y' + 2xy = 2x^3$.

$$4. \frac{dy}{dx} + \frac{2x}{1-x^2}y = 4x, \quad -1 < x < 1.$$

$$5. \frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4}{(1+x^2)^2}.$$

$$6. 2(\cos^2 x)y' + y \sin 2x = 4 \cos^4 x, \quad 0 \leq x < \pi/2.$$

$$7. y' + \frac{1}{x \ln x}y = 9x^2.$$

$$8. y' - y \tan x = 8 \sin^3 x.$$

$$9. t \frac{dx}{dt} + 2x = 4e^t, \quad t > 0.$$

$$10. y' = \sin x(y \sec x - 2).$$

$$11. (1 - y \sin x) dx - (\cos x) dy = 0.$$

$$12. y' - x^{-1}y = 2x^2 \ln x.$$

$$13. y' + \alpha y = e^{\beta x}, \text{ where } \alpha, \beta \text{ are constants.}$$

$$14. y' + mx^{-1}y = \ln x, \text{ where } m \text{ is constant.}$$

In Problems 15–20, solve the given initial-value problem.

$$15. y' + 2x^{-1}y = 4x, \quad y(1) = 2.$$

$$16. (\sin x)y' - y \cos x = \sin 2x, \quad y(\pi/2) = 2.$$

$$17. \frac{dx}{dt} + \frac{2}{4-t}x = 5, \quad x(0) = 4.$$

$$18. (y - e^x) dx + dy = 0, \quad y(0) = 1.$$

$$19. y' + y = f(x), \quad y(0) = 3, \text{ where}$$

$$f(x) = \begin{cases} 1, & \text{if } x \leq 1, \\ 0, & \text{if } x > 1. \end{cases}$$

$$20. y' - 2y = f(x), \quad y(0) = 1, \text{ where}$$

$$f(x) = \begin{cases} 1 - x, & \text{if } x < 1, \\ 0, & \text{if } x \geq 1. \end{cases}$$

21. Solve the initial-value problem in Example 1.6.5 as follows. First determine the general solution to the differential equation on each interval separately. Then use the given initial condition to find the appropriate integration constant for the interval $(-\infty, 1)$. To determine the integration constant on the interval $[1, \infty)$, use the fact that the solution must be continuous at $x = 1$.

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22. Find the general solution to the second-order differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} = 9x, \quad x > 0.$$

[Hint: Let $u = dy/dx$.]

23. Solve the differential equation for Newton’s law of cooling by viewing it as a first-order linear differential equation.

24. Suppose that an object is placed in a medium whose temperature is increasing at a constant rate of $\alpha^\circ\text{F}$ per minute. Show that, according to Newton’s law of cooling, the temperature of the object at time t is given by

$$T(t) = \alpha(t - k^{-1}) + c_1 + c_2e^{-kt},$$

where c_1 and c_2 are constants.

25. Between 8 a.m. and 12 p.m. on a hot summer day, the temperature rose at a rate of 10°F per hour from an initial temperature of 65°F . At 9 a.m. the temperature of an object was measured to be 35°F and was, at that time, increasing at a rate of 5°F per hour. Show that the temperature of the object at time t was

$$T(t) = 10t - 15 + 40e^{(1-t)/8}, \quad 0 \leq t \leq 4.$$

26. It is known that a certain object has constant of proportionality $k = 1/40$ in Newton’s law of cooling. When the temperature of this object is 0°F , it is placed in a medium whose temperature is changing in time according to

$$T_m(t) = 80e^{-t/20}.$$

- (a) Using Newton’s law of cooling, show that the temperature of the object at time t is

$$T(t) = 80(e^{-t/40} - e^{-t/20}).$$

- (b) What happens to the temperature of the object as $t \rightarrow +\infty$? Is this reasonable?
(c) Determine the time, t_{\max} , when the temperature of the object is a maximum. Find $T(t_{\max})$ and $T_m(t_{\max})$.
(d) Make a sketch to depict the behavior of $T(t)$ and $T_m(t)$.

27. The differential equation

$$\frac{dT}{dt} = -k_1[T - T_m(t)] + A_0, \quad (1.6.13)$$

where k_1 and A_0 are positive constants, can be used to model the temperature variation $T(t)$ in a building. In this equation, the first term on the right-hand side gives the contribution due to the variation in the outside temperature, and the second term on the right-hand side gives the contribution due to the heating effect from internal sources such as machinery, lighting, people, and so on. Consider the case when

$$T_m(t) = A - B \cos \omega t, \quad \omega = \pi/12, \quad (1.6.14)$$

where A and B are constants, and t is measured in hours.

- (a) Make a sketch of $T_m(t)$. Taking $t = 0$ to correspond to midnight, describe the variation of the external temperature over a 24-hour period.
(b) With T_m given in (1.6.14), solve (1.6.13) subject to the initial condition $T(0) = T_0$.

28. This problem demonstrates the **variation-of-parameters** method for first-order linear differential equations. Consider the first-order linear differential equation

$$y' + p(x)y = q(x). \quad (1.6.15)$$

- (a) Show that the general solution to the associated *homogeneous* equation

$$y' + p(x)y = 0$$

is

$$y_H(x) = c_1 e^{-\int p(x)dx}.$$

- (b) Determine the function $u(x)$ such that

$$y(x) = u(x)e^{-\int p(x)dx}$$

is a solution to (1.6.15), and hence derive the general solution to (1.6.15).

For Problems 29–32, use the technique derived in the previous problem to solve the given differential equation.

29. $y' + x^{-1}y = \cos x, \quad x > 0.$

30. $y' + y = e^{-2x}.$

31. $y' + y \cot x = 2 \cos x, \quad 0 < x < \pi.$

32. $xy' - y = x^2 \ln x.$

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For Problems 33–38, use a differential equation solver to determine the solution to each of the initial-value problems and sketch the corresponding solution curve.

33. ◇ The initial-value problem in Problem 15.

34. ◇ The initial-value problem in Problem 16.

35. ◇ The initial-value problem in Problem 17.

36. ◇ The initial-value problem in Problem 18.

37. ◇ The initial-value problem in Problem 19.

38. ◇ The initial-value problem in Problem 20.

1.7 Modeling Problems Using First-Order Linear Differential Equations

There are many examples of applied problems whose mathematical formulation leads to a first-order linear differential equation. In this section we analyze two in detail.

Mixing Problems

Statement of the Problem: Consider the situation depicted in Figure 1.7.1. A tank initially contains V_0 liters of a solution in which is dissolved A_0 grams of a certain chemical. A solution containing c_1 grams/liter of the same chemical flows into the tank at a constant rate of r_1 liters/minute, and the mixture flows out at a constant rate of r_2 liters/minute. We assume that the mixture is kept uniform by stirring. Then at any time t the concentration of chemical in the tank, $c_2(t)$, is the same throughout the tank and is given by

$$c_2 = \frac{A(t)}{V(t)}, \quad (1.7.1)$$

where $V(t)$ denotes the volume of solution in the tank at time t and $A(t)$ denotes the amount of chemical in the tank at time t .

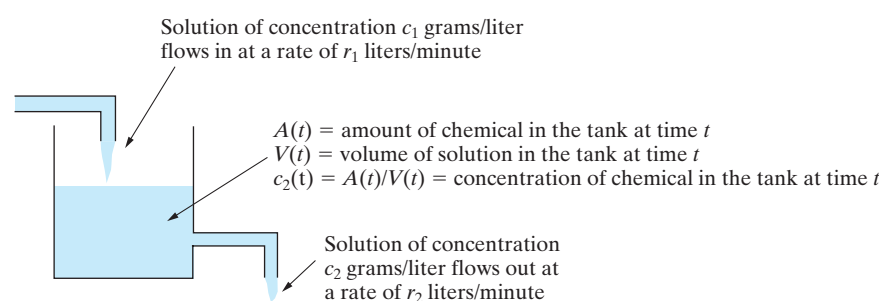


Figure 1.7.1: A mixing problem.

Mathematical Formulation: The two functions in the problem are $V(t)$ and $A(t)$. In order to determine how they change with time, we first consider their change during a short time interval, Δt minutes. In time Δt , $r_1 \Delta t$ liters of solution flow into the tank, whereas $r_2 \Delta t$ liters flow out. Thus during the time interval Δt , the *change* in the volume of solution in the tank is

$$\Delta V = r_1 \Delta t - r_2 \Delta t = (r_1 - r_2) \Delta t. \quad (1.7.2)$$

Since the concentration of chemical in the inflow is c_1 grams/liter (assumed constant), it follows that in the time interval Δt the amount of chemical that flows into the tank is $c_1 r_1 \Delta t$. Similarly, the amount of chemical that flows out in this same time interval is approximately⁶ $c_2 r_2 \Delta t$. Thus, the total change in the amount of chemical in the tank

⁶This is only an approximation, since c_2 is *not* constant over the time interval Δt . The approximation will become more accurate as $\Delta t \rightarrow 0$.