For Problems 33–38, use a differential equation solver to determine the solution to each of the initial-value problems and sketch the corresponding solution curve.

33. ◇ The initial-value problem in Problem 15.

34. ◇ The initial-value problem in Problem 16.

35. ◇ The initial-value problem in Problem 17.

36. ◇ The initial-value problem in Problem 18.

37. ◇ The initial-value problem in Problem 19.

38. ◇ The initial-value problem in Problem 20.

1.7 Modeling Problems Using First-Order Linear Differential Equations

There are many examples of applied problems whose mathematical formulation leads to a first-order linear differential equation. In this section we analyze two in detail.

Mixing Problems

Statement of the Problem: Consider the situation depicted in Figure 1.7.1. A tank initially contains V_0 liters of a solution in which is dissolved A_0 grams of a certain chemical. A solution containing c_1 grams/liter of the same chemical flows into the tank at a constant rate of r_1 liters/minute, and the mixture flows out at a constant rate of r_2 liters/minute. We assume that the mixture is kept uniform by stirring. Then at any time t the concentration of chemical in the tank, $c_2(t)$, is the same throughout the tank and is given by

$$c_2 = \frac{A(t)}{V(t)}, \qquad (1.7.1)$$

where $V(t)$ denotes the volume of solution in the tank at time t and $A(t)$ denotes the amount of chemical in the tank at time t.

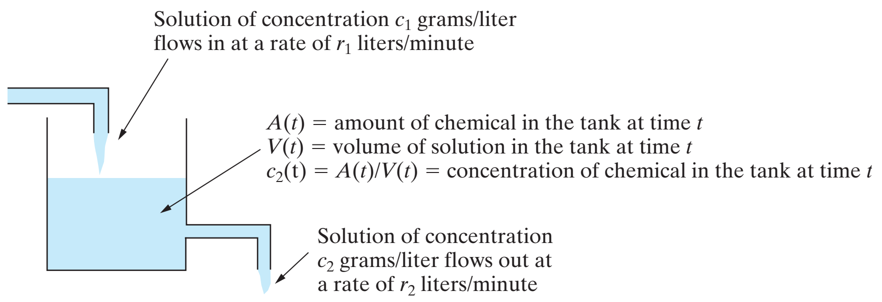

Figure 1.7.1: A mixing problem.

Mathematical Formulation: The two functions in the problem are $V(t)$ and $A(t)$. In order to determine how they change with time, we first consider their change during a short time interval, Δt minutes. In time Δt, $r_1\ \Delta t$ liters of solution flow into the tank, whereas $r_2\ \Delta t$ liters flow out. Thus during the time interval Δt, the *change* in the volume of solution in the tank is

$$\Delta V = r_1\ \Delta t - r_2\ \Delta t = (r_1 - r_2)\ \Delta t. \qquad (1.7.2)$$

Since the concentration of chemical in the inflow is c_1 grams/liter (assumed constant), it follows that in the time interval Δt the amount of chemical that flows into the tank is $c_1 r_1\ \Delta t$. Similarly, the amount of chemical that flows out in this same time interval is approximately[6] $c_2 r_2\ \Delta t$. Thus, the total change in the amount of chemical in the tank

[6]This is only an approximation, since c_2 is *not* constant over the time interval Δt. The approximation will become more accurate as $\Delta t \to 0$.

during the time interval Δt, denoted by ΔA, is approximately

$$\Delta A \approx c_1 r_1 \, \Delta t - c_2 r_2 \, \Delta t = (c_1 r_1 - c_2 r_2) \, \Delta t. \tag{1.7.3}$$

Dividing Equations (1.7.2) and (1.7.3) by Δt yields

$$\frac{\Delta V}{\Delta t} = r_1 - r_2 \quad \text{and} \quad \frac{\Delta A}{\Delta t} \approx c_1 r_1 - c_2 r_2,$$

respectively. These equations describe the rates of change of V and A over the short, but finite, time interval Δt. In order to determine the instantaneous rates of change of V and A, we take the limit as $\Delta t \to 0$ to obtain

$$\frac{dV}{dt} = r_1 - r_2 \tag{1.7.4}$$

and

$$\frac{dA}{dt} = c_1 r_1 - \frac{A}{V} r_2, \tag{1.7.5}$$

where we have substituted for c_2 from Equation (1.7.1). Since r_1 and r_2 are constants, we can integrate Equation (1.7.4) directly, obtaining

$$V(t) = (r_1 - r_2)t + V_0,$$

where V_0 is an integration constant. Substituting for V into Equation (1.7.5) and rearranging terms yields the linear equation for $A(t)$:

$$\frac{dA}{dt} + \frac{r_2}{(r_1 - r_2)t + V_0} A = c_1 r_1. \tag{1.7.6}$$

This differential equation can be solved, subject to the initial condition $A(0) = A_0$, to determine the behavior of $A(t)$.

Remark The reader need not memorize Equation (1.7.6), since it is better to derive it for each specific example.

Example 1.7.1 A tank contains 8 L (liters) of water in which is dissolved 32 g (grams) of chemical. A solution containing 2 g/L of the chemical flows into the tank at a rate of 4 L/min, and the well-stirred mixture flows out at a rate of 2 L/min.

1. Determine the amount of chemical in the tank after 20 minutes.
2. What is the concentration of chemical in the tank at that time?

Solution: We are given

$$r_1 = 4 \text{ L/min}, \quad r_2 = 2 \text{ L/min}, \quad c_1 = 2 \text{ g/L}, \quad V(0) = 8 \text{ L}, \quad \text{and} \quad A(0) = 32 \text{ g}.$$

For parts 1 and 2, we must find $A(20)$ and $A(20)/V(20)$, respectively. Now,

$$\Delta V = r_1 \, \Delta t - r_2 \, \Delta t$$

implies that

$$\frac{dV}{dt} = 2.$$

Integrating this equation and imposing the initial condition that $V(0) = 8$ yields

$$V(t) = 2(t + 4). \tag{1.7.7}$$

Further,

$$\Delta A \approx c_1 r_1 \,\Delta t - c_2 r_2 \,\Delta t$$

implies that

$$\frac{dA}{dt} = 8 - 2c_2.$$

That is, since $c_2 = A/V$,

$$\frac{dA}{dt} = 8 - 2\frac{A}{V}.$$

Substituting for V from (1.7.7), we must solve

$$\frac{dA}{dt} + \frac{1}{t+4}A = 8. \tag{1.7.8}$$

This first-order linear equation has integrating factor

$$I = e^{\int 1/(t+4)dt} = t + 4.$$

Consequently (1.7.8) can be written in the equivalent form

$$\frac{d}{dt}[(t+4)A] = 8(t+4),$$

which can be integrated directly to obtain

$$(t+4)A = 4(t+4)^2 + c.$$

Hence

$$A(t) = \frac{1}{t+4}[4(t+4)^2 + c].$$

Imposing the given initial condition $A(0) = 32$ g implies that $c = 64$. Consequently

$$A(t) = \frac{4}{t+4}[(t+4)^2 + 16].$$

Setting $t = 20$ gives us the values for parts 1 and 2:

1. We have
$$A(20) = \frac{1}{6}[(24)^2 + 16] = \frac{296}{3} \text{ g}.$$

2. Furthermore, using (1.7.7),
$$\frac{A(20)}{V(20)} = \frac{1}{48} \cdot \frac{296}{3} = \frac{37}{18} \text{ g/L}.$$ □

Electric Circuits

An important application of differential equations arises from the analysis of simple electric circuits. The most basic electric circuit is obtained by connecting the ends of a wire to the terminals of a battery or generator. This causes a flow of charge, $q(t)$, measured in coulombs (C), through the wire, thereby producing a current, $i(t)$, measured in amperes (A), defined to be the rate of change of charge. Thus,

$$i(t) = \frac{dq}{dt}. \tag{1.7.9}$$

In practice a circuit will contain several components that oppose the flow of charge. As current passes through these components, work has to be done, and the loss of energy is described by the resulting voltage drop across each component. For the circuits that we will consider, the behavior of the current in the circuit is governed by Kirchoff's second law, which can be stated as follows.

Kirchoff's Second Law: The sum of the voltage drops around a closed circuit is zero.

In order to apply this law we need to know the relationship between the current passing through each component in the circuit and the resulting voltage drop. The components of interest to us are resistors, capacitors, and inductors. We briefly describe each of these next.

1. *Resistors*: A resistor is a component that, owing to its constituency, directly resists the flow of charge through it. According to *Ohm's law*, the voltage drop, ΔV_R, between the ends of a resistor is directly proportional to the current that is passing through it. This is expressed mathematically as

$$\Delta V_R = iR \tag{1.7.10}$$

   where the constant of proportionality, R, is called the **resistance** of the resistor. The units of resistance are ohms (Ω).

2. *Capacitors*: A capacitor can be thought of as a component that stores charge and thereby opposes the passage of current. If $q(t)$ denotes the charge on the capacitor at time t, then the drop in voltage, ΔV_C, as current passes through it is directly proportional to $q(t)$. It is usual to express this law in the form

$$\Delta V_C = \frac{1}{C}q, \tag{1.7.11}$$

   where the constant C is called the **capacitance** of the capacitor. The units of capacitance are farads (F).

3. *Inductors*: The third component that is of interest to us is an inductor. This can be considered as a component that opposes any change in the current flowing through it. The drop in voltage as current passes through an inductor is directly proportional to the rate at which the current is changing. We write this as

$$\Delta V_L = L\frac{di}{dt}, \tag{1.7.12}$$

   where the constant L is called the **inductance** of the inductor, measured in units of henrys (H).

4. *EMF*: The final component in our circuits will be a source of voltage that produces an electromotive force (EMF), driving the charge through the circuit. As current passes through the voltage source, there is a voltage gain, which we denote by $E(t)$ volts (that is, a voltage drop of $-E(t)$ volts).

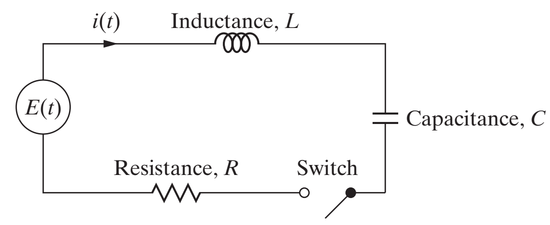

Figure 1.7.2: A simple RLC circuit.

A circuit containing all of these components is shown in Figure 1.7.2. Such a circuit is called an **RLC circuit**. According to Kirchoff's second law, the sum of the voltage drops at any instant must be zero. Applying this to the RLC circuit in Figure 1.7.2, we obtain

$$\Delta V_R + \Delta V_C + \Delta V_L - E(t) = 0. \tag{1.7.13}$$

Substituting into Equation (1.7.13) from (1.7.10)–(1.7.12) and rearranging yields the basic differential equation for an RLC circuit—namely,

$$L\frac{di}{dt} + Ri + \frac{q}{C} = E(t). \tag{1.7.14}$$

Three cases are important in applications, two of which are governed by first-order linear differential equations.

Case 1: An RL CIRCUIT. In the case when no capacitor is present, we have what is referred to as an RL circuit. The differential equation (1.7.14) then reduces to

$$\frac{di}{dt} + \frac{R}{L}i = \frac{1}{L}E(t). \tag{1.7.15}$$

This is a first-order linear differential equation for the current in the circuit at any time t.

Case 2: An RC CIRCUIT. Now consider the case when no inductor is present in the circuit. Setting $L = 0$ in Equation (1.7.14) yields

$$i + \frac{1}{RC}q = \frac{E}{R}.$$

In this equation we have two unknowns, $q(t)$ and $i(t)$. Substituting from (1.7.9) for $i(t) = dq/dt$, we obtain the following differential equation for $q(t)$:

$$\frac{dq}{dt} + \frac{1}{RC}q = \frac{E}{R}. \tag{1.7.16}$$

In this case, the first-order linear differential equation (1.7.16) can be solved for the charge $q(t)$ on the plates of the capacitor. The current in the circuit can then be obtained from

$$i(t) = \frac{dq}{dt}$$

by differentiation.

Case 3: An RLC CIRCUIT. In the general case, we must consider all three components to be present in the circuit. Substituting from Equation (1.7.9) into Equation (1.7.14) yields the following differential equation for determining the charge on the capacitor:

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{1}{L}E(t).$$

We will develop techniques in Chapter 6 that enable us to solve this differential equation without difficulty.

For the remainder of this section we restrict our attention to RL and RC circuits.Since these are both first-order linear differential equations, we can solve them using the technique derived in the previous section, once the applied EMF, $E(t)$, has been specified. The two most important forms for $E(t)$ are

$$E(t) = E_0 \qquad \text{and} \qquad E(t) = E_0 \cos \omega t,$$

where E_0 and ω are constants. The first of these corresponds to a source of EMF such as a battery. The resulting current is called a **direct current** (DC). The second form of EMF oscillates between $\pm E_0$ and is called an **alternating current** (AC).

Example 1.7.2 Determine the current in an RL circuit if the applied EMF is $E(t) = E_0 \cos \omega t$, where E_0 and ω are constants, and the initial current is zero.

Solution: Substituting into Equation (1.7.15) for $E(t)$ yields the differential equation

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E_0}{L}\cos \omega t,$$

which we write as

$$\frac{di}{dt} + ai = \frac{E_0}{L}\cos \omega t, \tag{1.7.17}$$

where $a = R/L$. An integrating factor for (1.7.17) is $I(t) = e^{at}$, so that the equation can be written in the equivalent form

$$\frac{d}{dt}(e^{at}i) = \frac{E_0}{L}e^{at}\cos \omega t.$$

Integrating this equation using the standard integral

$$\int e^{at}\cos \omega t \, dt = \frac{1}{a^2+\omega^2}e^{at}(a\cos \omega t + \omega \sin \omega t) + c,$$

we obtain

$$e^{at}i = \frac{E_0}{L(a^2+\omega^2)}e^{at}(a\cos \omega t + \omega \sin \omega t) + c,$$

where c is an integration constant. Consequently,

$$i(t) = \frac{E_0}{L(a^2+\omega^2)}(a\cos \omega t + \omega \sin \omega t) + ce^{-at}.$$

Imposing the initial condition $i(0) = 0$, we find

$$c = -\frac{E_0 a}{L(a^2+\omega^2)},$$

so that

$$i(t) = \frac{E_0}{L(a^2+\omega^2)}(a\cos\omega t + \omega \sin\omega t - ae^{-at}). \tag{1.7.18}$$

This solution can be written in the form

$$i(t) = i_S(t) + i_T(t),$$

where

$$i_S(t) = \frac{E_0}{L(a^2+\omega^2)}(a\cos\omega t + \omega\sin\omega t), \qquad i_T(t) = -\frac{aE_0}{L(a^2+\omega^2)}e^{-at}.$$

The term $i_T(t)$ decays exponentially with time and is referred to as the **transient part** of the solution. As $t \to \infty$, the solution (1.7.18) approaches the **steady-state solution**, $i_S(t)$. The steady-state solution can be written in a more illuminating form as follows. If we construct the right-angled triangle (see Figure 1.7.3) with sides a and ω, then the hypotenuse of the triangle is $\sqrt{a^2+\omega^2}$. Consequently, there exists a unique angle ϕ in $(0, \pi/2)$, such that

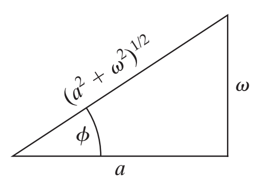

Figure 1.7.3: Defining the phase angle for an RL circuit.

$$\cos\phi = \frac{a}{\sqrt{a^2+\omega^2}}, \qquad \sin\phi = \frac{\omega}{\sqrt{a^2+\omega^2}}.$$

Equivalently,

$$a = \sqrt{a^2+\omega^2}\cos\phi, \qquad \omega = \sqrt{a^2+\omega^2}\sin\phi.$$

Substituting for a and ω into the expression for i_S yields

$$i_S(t) = \frac{E_0}{L\sqrt{a^2+\omega^2}}(\cos\omega t\cos\phi + \sin\omega t \sin\phi),$$

which can be written, using an appropriate trigonometric identity, as

$$i_S(t) = \frac{E_0}{L\sqrt{a^2+\omega^2}}\cos(\omega t - \phi).$$

This is referred to as the phase-amplitude form of the solution. Comparing this with the original driving term, $E_0 \cos\omega t$, we see that the system has responded with a steady-state solution having the same periodic behavior, but with a phase shift of ϕ radians. Furthermore the amplitude of the response is

$$A = \frac{E_0}{L\sqrt{a^2+\omega^2}} = \frac{E_0}{\sqrt{R^2+\omega^2L^2}}, \tag{1.7.19}$$

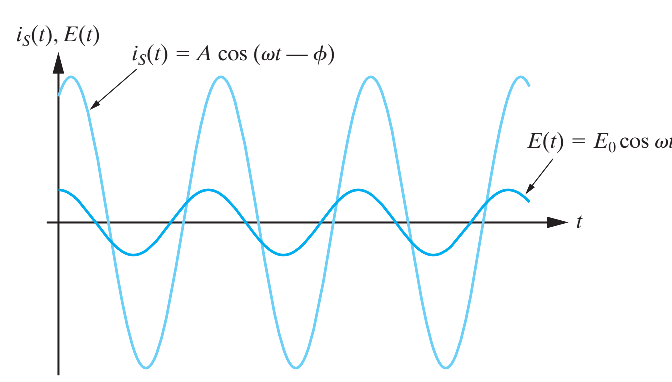

Figure 1.7.4: The response of an RL circuit to the driving term $E(t) = E_0 \cos\omega t$.

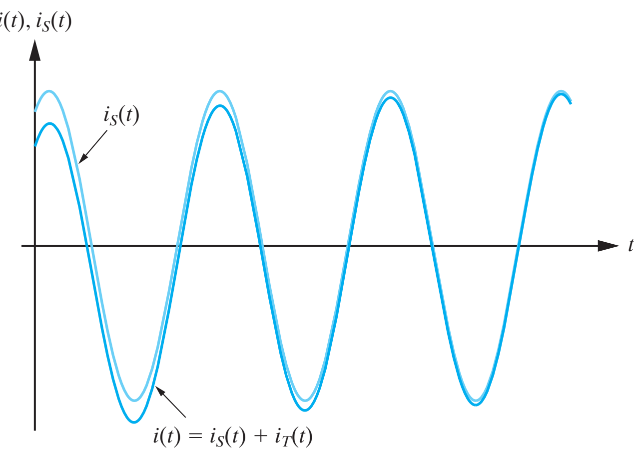

Figure 1.7.5: The transient part of the solution for an RL circuit dies out as t increases.

where we have substituted for $a = R/L$. This is illustrated in Figure 1.7.4. The general picture that we have, therefore, is that the transient part of the solution affects $i(t)$ for a short period of time, after which the current settles into a steady-state. In the case when the driving EMF has the form $E(t) = E_0 \cos \omega t$, the steady-state is a phase shift of this driving EMF with an amplitude given in Equation (1.7.19). This general behavior is illustrated in Figure 1.7.5. □

Our next example illustrates the procedure for solving the differential equation (1.7.16) governing the behavior of an RC circuit.

Example 1.7.3 Consider the RC circuit in which $R = 0.5\ \Omega$, $C = 0.1$ F, and $E_0 = 20$ V. Given that the capacitor has zero initial charge, determine the current in the circuit after 0.25 seconds.

Solution: In this case we first solve Equation (1.7.16) for $q(t)$ and then determine the current in the circuit by differentiating the result. Substituting for R, C and E into Equation (1.7.16) yields

$$\frac{dq}{dt} + 20q = 40,$$

which has general solution

$$q(t) = 2 + ce^{-20t},$$

where c is an integration constant. Imposing the initial condition $q(0) = 0$ yields $c = -2$, so that

$$q(t) = 2(1 - e^{-20t}).$$

Differentiating this expression for q gives the current in the circuit

$$i(t) = \frac{dq}{dt} = 40e^{-20t}.$$

Consequently,

$$i(0.25) = 40e^{-5} \approx 0.27 \text{ A}.$$ □

Exercises for 1.7

Key Terms

Mixing problem, Concentration, Electric circuit, Kirchoff's second law, Resistor, Capacitor, Inductor, Electromotive force (EMF), RL circuit, RC circuit, RLC circuit, Direct current, Alternating current, Transient solution, Steady-state solution, Phase, Amplitude.

Skills

- Be able to use information about a mixing problem to provide the correct mathematical formulation of the problem.
- Be able to solve mixing problems by deriving and solving the differential equation (1.7.6) for a specific mixing problem and using initial conditions.
- Know the relationship between the charge and the current in an electric circuit.
- Be familiar with the basic components of an electric circuit, such as electromotive force, resistors, capacitors, and inductors.
- Be able to write down and solve the differential equation for the current in an RL circuit and for the charge in an RC circuit, for either a direct current or an alternating current.
- Be able to identify the transient and steady-state components of current in an electric circuit with an alternating current.
- Be able to put the steady-state component of the current in an RL circuit in phase-amplitude form, and identify the phase shift and the amplitude.

True-False Review

For Questions 1–8, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. The amount of chemical $A(t)$ in a tank at time t is obtained by multiplying the concentration of chemical $c(t)$ in the tank at time t by the volume of the solution, $V(t)$, at time t.

2. If r_1 and r_2 denote the rates at which fluid is flowing into a tank and out of the tank, respectively, then the rate of change of the volume of the tank is $r_2 - r_1$.

3. For the mixing problems described in this section, we assume that the concentration of the chemical entering the tank is independent of time.

4. For the mixing problems described in this section, we assume that the concentration of the chemical leaving the tank is independent of time.

5. Kirchoff's second law states the sum of the voltage drops around a closed circuit is independent of time.

6. The larger the resistance in a resistor, the greater the voltage drop between the ends of the resistor.

7. Given an alternating current in an RL circuit, the transient part of the current decays to zero with time, while the steady-state part of the current oscillates with the same frequency as the applied EMF.

8. The higher the frequency of an applied EMF in an RL circuit, the lower the amplitude of the steady-state current.

Problems

1. A container initially contains 10 L of water in which there is 20 g of salt dissolved. A solution containing 4 g/L of salt is pumped into the container at a rate of 2 L/min, and the well-stirred mixture runs out at a rate of 1 L/min. How much salt is in the tank after 40 minutes?

2. A tank initially contains 600 L of solution in which there is dissolved 1500 g of chemical. A solution containing 5 g/L of the chemical flows into the tank at a rate of 6 L/min, and the well-stirred mixture flows out at a rate of 3 L/min. Determine the concentration of chemical in the tank after one hour.

3. A tank whose volume is 40 L initially contains 20 L of water. A solution containing 10 g/L of salt is pumped into the tank at a rate of 4 L/min, and the well-stirred mixture flows out at a rate of 2 L/min. How much salt is in the tank just before the solution overflows?

4. A tank whose volume is 200 L is initially half full of a solution that contains 100 g of chemical. A solution containing 0.5 g/L of the same chemical flows into the tank at a rate of 6 L/min, and the well-stirred mixture flows out at a rate of 4 L/min. Determine the concentration of chemical in the tank just before the solution overflows.

5. A tank initially contains 10 L of a salt solution. Water flows into the tank at a rate of 3 L/min, and the well-stirred mixture flows out at a rate of 2 L/min. After 5 min, the concentration of salt in the tank is 0.2 g/L. Find:

(a) The amount of salt in the tank initially.

(b) The volume of solution in the tank when the concentration of salt is 0.1 g/L.

6. A tank initially contains 20 L of water. A solution containing 1 g/L of chemical flows into the tank at a rate of 3 L/min, and the mixture flows out at a rate of 2 L/min.

(a) Set up and solve the initial-value problem for $A(t)$, the amount of chemical in the tank at time t.

(b) When does the concentration of chemical in the tank reach 0.5 g/L?

7. A tank initially contains w liters of a solution in which is dissolved A_0 grams of chemical. A solution containing k g/L of this chemical flows into the tank at a rate of r L/min, and the mixture flows out at the same rate.

(a) Show that the amount of chemical, $A(t)$, in the tank at time t is

$$A(t) = e^{-(rt)/w}[kw(e^{(rt)/w} - 1) + A_0].$$

(b) Show that as $t \to \infty$, the concentration of chemical in the tank approaches k g/L. Is this result reasonable? Explain.

8. Consider the double mixing problem depicted in Figure 1.7.6.

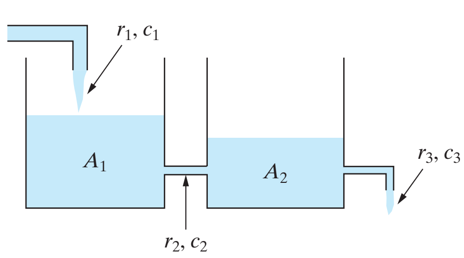

Figure 1.7.6: Double mixing problem

(a) Show that the following are differential equations for $A_1(t)$ and $A_2(t)$:

$$\frac{dA_1}{dt} + \frac{r_2}{(r_1 - r_2)t + V_1} A_1 = c_1 r_1,$$

$$\frac{dA_2}{dt} + \frac{r_3}{(r_2 - r_3)t + V_2} A_2 = \frac{r_2 A_1}{(r_1 - r_2)t + V_1},$$

where V_1 and V_2 are constants.

(b) Let $r_1 = 6$ L/min, $r_2 = 4$ L/min, $r_3 = 3$ L/min, and $c_1 = 0.5$ g/L. If the first tank initially holds 40 L of water in which 4 grams of chemical is dissolved, whereas the second tank initially contains 20 g of chemical dissolved in 20 L of water, determine the amount of chemical in the second tank after 10 min.

9. Consider the RL circuit in which $R = 4\ \Omega$, $L = 0.1$ H, and $E(t) = 20$ V. If no current is flowing initially, determine the current in the circuit for $t \geq 0$.

10. Consider the RC circuit which has $R = 5\ \Omega$, $C = \frac{1}{50}$ F, and $E(t) = 100$ V. If the capacitor is uncharged initially, determine the current in the circuit for $t \geq 0$.

11. An RL circuit has EMF $E(t) = 10 \sin 4t$ V. If $R = 2\ \Omega$, $L = \frac{2}{3}$ H, and there is no current flowing initially, determine the current for $t \geq 0$.

12. Consider the RC circuit with $R = 2\ \Omega$, $C = \frac{1}{8}$ F, and $E(t) = 10 \cos 3t$ V. If $q(0) = 1$ C, determine the current in the circuit for $t \geq 0$.

13. Consider the general RC circuit with $E(t) = 0$. Suppose that $q(0) = 5$ C. Determine the charge on the capacitor for $t > 0$. What happens as $t \to \infty$? Is this reasonable? Explain.

14. Determine the current in an RC circuit if the capacitor has zero charge initially and the driving EMF is $E = E_0$, where E_0 is a constant. Make a sketch showing the change in the charge $q(t)$ on the capacitor with time and show that $q(t)$ approaches a constant value as $t \to \infty$. What happens to the current in the circuit as $t \to \infty$?

15. Determine the current flowing in an RL circuit if the applied EMF is $E(t) = E_0 \sin \omega t$, where E_0 and ω are constants. Identify the transient part of the solution and the steady-state solution.

16. Determine the current flowing in an RL circuit if the applied EMF is constant and the initial current is zero.

17. Determine the current flowing in an RC circuit if the capacitor is initially uncharged and the driving EMF is given by $E(t) = E_0 e^{-at}$, where E_0 and a are constants.

18. Consider the special case of the RLC circuit in which the resistance is negligible and the driving EMF is zero. The differential equation governing the charge on the capacitor in this case is

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0.$$

If the capacitor has an initial charge of q_0 coulombs, and no current is flowing initially, determine the charge on the capacitor for $t > 0$, and the corresponding current in the circuit. [**Hint:** Let $u = dq/dt$ and use the chain rule to show that this implies $du/dt = u(du/dq)$.]

19. Repeat the previous problem for the case in which the driving EMF is $E(t) = E_0$, a constant.

1.8 Change of Variables

So far we have introduced techniques for solving separable and first-order linear differential equations. Clearly, most first-order differential equations are not of these two types. In this section, we consider two further types of differential equations that can be solved by using a change of variables to reduce them to one of the types we know how to solve. The key point to grasp, however, is not the specific changes of variables that we discuss, but the general idea of changing variables in a differential equation. Further examples are considered in the exercises. We first require a preliminary definition.

DEFINITION 1.8.1

A function $f(x, y)$ is said to be **homogeneous of degree zero**[7] if

$$f(tx, ty) = f(x, y)$$

for all positive values of t for which (tx, ty) is in the domain of f.

Remark Equivalently, we can say that f is homogeneous of degree zero if it is invariant under a rescaling of the variables x and y.

The simplest nonconstant functions that are homogeneous of degree zero are $f(x, y) = y/x$, and $f(x, y) = x/y$.

Example 1.8.2 If $f(x, y) = \dfrac{x^2 - y^2}{2xy + y^2}$, then

$$f(tx, ty) = \frac{t^2(x^2 - y^2)}{t^2(2xy + y^2)} = f(x, y),$$

so that f is homogeneous of degree zero. □

In the previous example, if we factor an x^2 term from the numerator and denominator, then the function f can be written in the form

$$f(x, y) = \frac{x^2[1 - (y/x)^2]}{x^2[2(y/x) + (y/x)^2]}.$$

That is,

$$f(x, y) = \frac{1 - (y/x)^2}{2(y/x) + (y/x)^2}.$$

[7] More generally, $f(x, y)$ is said to be **homogeneous of degree** m if $f(tx, ty) = t^m f(x, y)$.