18. Consider the special case of the RLC circuit in which the resistance is negligible and the driving EMF is zero. The differential equation governing the charge on the capacitor in this case is

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0.$$

If the capacitor has an initial charge of q_0 coulombs, and no current is flowing initially, determine the charge on the capacitor for $t > 0$, and the corresponding current in the circuit. [**Hint:** Let $u = dq/dt$ and use the chain rule to show that this implies $du/dt = u(du/dq)$.]

19. Repeat the previous problem for the case in which the driving EMF is $E(t) = E_0$, a constant.

1.8 Change of Variables

So far we have introduced techniques for solving separable and first-order linear differential equations. Clearly, most first-order differential equations are not of these two types. In this section, we consider two further types of differential equations that can be solved by using a change of variables to reduce them to one of the types we know how to solve. The key point to grasp, however, is not the specific changes of variables that we discuss, but the general idea of changing variables in a differential equation. Further examples are considered in the exercises. We first require a preliminary definition.

DEFINITION 1.8.1

A function $f(x, y)$ is said to be **homogeneous of degree zero**[7] if

$$f(tx, ty) = f(x, y)$$

for all positive values of t for which (tx, ty) is in the domain of f.

Remark Equivalently, we can say that f is homogeneous of degree zero if it is invariant under a rescaling of the variables x and y.

The simplest nonconstant functions that are homogeneous of degree zero are $f(x, y) = y/x$, and $f(x, y) = x/y$.

Example 1.8.2 If $f(x, y) = \dfrac{x^2 - y^2}{2xy + y^2}$, then

$$f(tx, ty) = \frac{t^2(x^2 - y^2)}{t^2(2xy + y^2)} = f(x, y),$$

so that f is homogeneous of degree zero. □

In the previous example, if we factor an x^2 term from the numerator and denominator, then the function f can be written in the form

$$f(x, y) = \frac{x^2[1 - (y/x)^2]}{x^2[2(y/x) + (y/x)^2]}.$$

That is,

$$f(x, y) = \frac{1 - (y/x)^2}{2(y/x) + (y/x)^2}.$$

[7] More generally, $f(x, y)$ is said to be **homogeneous of degree** m if $f(tx, ty) = t^m f(x, y)$.

Thus f can be considered to depend on the single variable $V = y/x$. The following theorem establishes that this is a basic property of all functions that are homogeneous of degree zero.

Theorem 1.8.3 A function $f(x, y)$ is homogeneous of degree zero if and only if it depends on y/x only.

Proof Suppose that f is homogeneous of degree zero. We must consider two cases separately.

(a) If $x > 0$, we can take $t = 1/x$ in Definition 1.8.1 to obtain

$$f(x, y) = f(1, y/x),$$

which is a function of $V = y/x$ only.

(b) If $x < 0$, then we can take $t = -1/x$ in Definition 1.8.1. In this case we obtain

$$f(x, y) = f(-1, -y/x),$$

which once more depends on y/x only.

Conversely, suppose that $f(x, y)$ depends only on y/x. If we replace x by tx and y by ty, then f is unaltered, since $y/x = (ty)/(tx)$, and hence is homogeneous of degree zero. ■

Remark Do not memorize the formulas in the preceding theorem. Just remember that a function $f(x, y)$ that is homogeneous of degree zero depends only on the combination y/x and hence can be considered as a function of a single variable, say, V, where $V = y/x$.

We now consider solving differential equations that satisfy the following definition.

DEFINITION 1.8.4

If $f(x, y)$ is homogeneous of degree zero, then the differential equation

$$\frac{dy}{dx} = f(x, y)$$

is called a **homogeneous first-order differential equation**.

In general, if

$$\frac{dy}{dx} = f(x, y)$$

is a homogeneous first-order differential equation, then we cannot solve it directly. However, our preceding discussion implies that such a differential equation can be written in the equivalent form

$$\frac{dy}{dx} = F(y/x), \tag{1.8.1}$$

for an appropriate function F. This suggests that, instead of using the variables x and y, we should use the variables x and V, where $V = y/x$, or equivalently,

$$y = xV(x). \tag{1.8.2}$$

Substitution of (1.8.2) into the right-hand side of Equation (1.8.1) has the effect of reducing it to a function of V only. We must also determine how the derivative term dy/dx transforms. Differentiating (1.8.2) with respect to x using the product rule yields the following relationship between dy/dx and dV/dx:

$$\frac{dy}{dx} = x\frac{dV}{dx} + V.$$

Substituting into Equation (1.8.1), we therefore obtain

$$x\frac{dV}{dx} + V = F(V),$$

or equivalently,

$$x\frac{dV}{dx} = F(V) - V.$$

The variables can now be separated to yield

$$\frac{1}{F(V) - V}\,dV = \frac{1}{x}\,dx,$$

which can be solved directly by integration. We have therefore established the next theorem.

Theorem 1.8.5 The change of variables $y = xV(x)$ reduces a homogeneous first-order differential equation $dy/dx = f(x, y)$ to the separable equation

$$\frac{1}{F(V) - V}\,dV = \frac{1}{x}\,dx.$$

Remark The separable equation that results in the previous technique can be integrated to obtain a relationship between V and x. We then obtain the solution to the given differential equation by substituting y/x for V in this relationship.

Example 1.8.6 Find the general solution to

$$\frac{dy}{dx} = \frac{4x + y}{x - 4y}. \tag{1.8.3}$$

Solution: The function on the right-hand side of Equation (1.8.3) is homogeneous of degree zero, so that we have a first-order homogeneous differential equation. Substituting $y = xV$ into the equation yields

$$\frac{d}{dx}(xV) = \frac{4 + V}{1 - 4V}.$$

That is,

$$x\frac{dV}{dx} + V = \frac{4 + V}{1 - 4V},$$

or equivalently,

$$x\frac{dV}{dx} = \frac{4(1+V^2)}{1-4V}.$$

Separating the variables gives

$$\frac{1-4V}{4(1+V^2)}\,dV = \frac{1}{x}\,dx.$$

We write this as

$$\left[\frac{1}{4(1+V^2)} - \frac{V}{1+V^2}\right] dV = \frac{1}{x}\,dx,$$

which can be integrated directly to obtain

$$\frac{1}{4}\arctan V - \frac{1}{2}\ln\,(1+V^2) = \ln|x| + c.$$

Substituting $V = y/x$ and multiplying through by 2 yields

$$\frac{1}{2}\arctan\left(\frac{y}{x}\right) - \ln\left(\frac{x^2+y^2}{x^2}\right) = \ln\,(x^2) + c_1,$$

which simplifies to

$$\frac{1}{2}\arctan\left(\frac{y}{x}\right) - \ln\,(x^2+y^2) = c_1. \tag{1.8.4}$$

Although this technically gives the answer, the solution is more easily expressed in terms of polar coordinates:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \quad \Longleftrightarrow \quad r = \sqrt{x^2+y^2} \quad \text{and} \quad \theta = \arctan\left(\frac{y}{x}\right).$$

Substituting into Equation (1.8.4) yields

$$\frac{1}{2}\theta - \ln\,(r^2) = c_1,$$

or equivalently,

$$\ln r = \frac{1}{4}\theta + c_2.$$

Exponentiating both sides of this equation gives

$$r = c_3 e^{\theta/4}.$$

For each value of c_3, this is the equation of a logarithmic spiral. The particular spiral with equation $r = \frac{1}{2}e^{\theta/4}$ is shown in Figure 1.8.1.

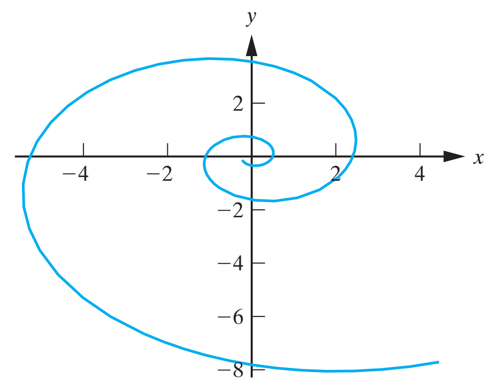

Figure 1.8.1: Graph of the logarithmic spiral with polar equation $r = \frac{1}{2}e^{\theta/4}$, $-5\pi/6 \leq \theta \leq 22\pi/6$.

□

Example 1.8.7 Find the equation of the orthogonal trajectories to the family

$$x^2 + y^2 - 2cx = 0. \tag{1.8.5}$$

(Completing the square in x, we obtain $(x-c)^2 + y^2 = c^2$, which represents the family of circles centered at $(c, 0)$, with radius c.)

Solution: First we need an expression for the slope of the given family at the point (x, y). Differentiating Equation (1.8.5) implicitly with respect to x yields

$$2x + 2y\frac{dy}{dx} - 2c = 0,$$

which simplifies to

$$\frac{dy}{dx} = \frac{c-x}{y}. \tag{1.8.6}$$

This is not the differential equation of the given family, since it still contains the constant c and hence is dependent on the individual curves in the family. Therefore, we must eliminate c to obtain an expression for the slope of the family that is independent of any particular curve in the family. From Equation (1.8.5) we have

$$c = \frac{x^2+y^2}{2x}.$$

Substituting this expression for c into Equation (1.8.6) and simplifying gives

$$\frac{dy}{dx} = \frac{y^2-x^2}{2xy}.$$

Therefore, the differential equation for the family of orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{2xy}{y^2-x^2}. \tag{1.8.7}$$

This differential equation is first-order homogeneous. Substituting $y = xV(x)$ into Equation (1.8.7) yields

$$\frac{d}{dx}(xV) = \frac{2V}{1-V^2},$$

so that

$$x\frac{dV}{dx} + V = \frac{2V}{1-V^2}.$$

Hence

$$x\frac{dV}{dx} = \frac{V+V^3}{1-V^2},$$

or in separated form,

$$\frac{1-V^2}{V(1+V^2)}\,dV = \frac{1}{x}\,dx.$$

Decomposing the left-hand side into partial fractions yields

$$\left(\frac{1}{V} - \frac{2V}{1+V^2}\right) dV = \frac{1}{x}\,dx,$$

which can be integrated directly to obtain

$$\ln|V| - \ln\,(1+V^2) = \ln|x| + c,$$

or equivalently,

$$\ln\left(\frac{|V|}{1+V^2}\right) = \ln|x| + c.$$

Exponentiating both sides and redefining the constant yields

$$\frac{V}{1+V^2} = c_1 x.$$

Substituting back for $V = y/x$, we obtain

$$\frac{xy}{x^2+y^2} = c_1 x.$$

That is,

$$x^2 + y^2 = c_2 y,$$

where $c_2 = 1/c_1$. Completing the square in y yields

$$x^2 + (y-k)^2 = k^2, \tag{1.8.8}$$

where $k = c_2/2$. Equation (1.8.8) is the equation of the family of orthogonal trajectories. This is the family of circles centered at $(0, k)$ with radius k (circles along the y-axis). (See Figure 1.8.2.)

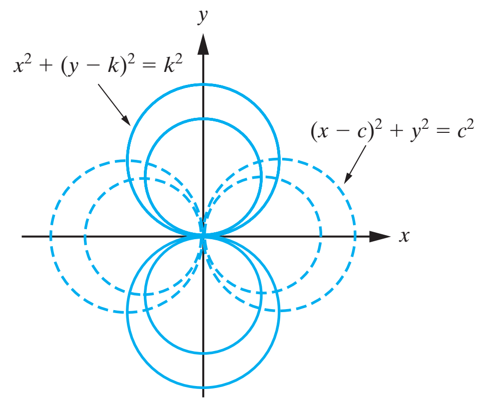

Figure 1.8.2: The family $(x - c)^2 + y^2 = c^2$ and its orthogonal trajectories $x^2 + (y - k)^2 = k^2$.

□

Bernoulli Equations

We now consider a special type of nonlinear differential equation that can be reduced to a linear equation by a change of variables.

DEFINITION 1.8.8

A differential equation that can be written in the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \tag{1.8.9}$$

where n is a real constant, is called a **Bernoulli equation**.

If $n = 0$ or $n = 1$, Equation (1.8.9) is linear, but otherwise it is nonlinear. We can reduce it to a linear equation as follows. We first divide Equation (1.8.9) by y^n to obtain

$$y^{-n}\frac{dy}{dx} + y^{1-n}p(x) = q(x). \tag{1.8.10}$$

We now make the change of variables

$$u(x) = y^{1-n}, \tag{1.8.11}$$

which implies that

$$\frac{du}{dx} = (1 - n)y^{-n}\frac{dy}{dx}.$$

That is,

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx}.$$

Substituting into Equation (1.8.10) for y^{1-n} and $y^{-n}dy/dx$ yields the linear differential equation

$$\frac{1}{1-n}\frac{du}{dx} + p(x)u = q(x),$$

or in standard form,

$$\frac{du}{dx} + (1-n)p(x)u = (1-n)q(x). \tag{1.8.12}$$

The linear equation (1.8.12) can now be solved for u as a function of x. The solution to the original equation is then obtained from (1.8.11).

Example 1.8.9 Solve

$$\frac{dy}{dx} + \frac{3}{x}y = \frac{12y^{2/3}}{\sqrt{1+x^2}}, \qquad x > 0.$$

Solution: The differential equation is a Bernoulli equation. Dividing both sides of the differential equation by $y^{2/3}$ yields

$$y^{-2/3}\frac{dy}{dx} + \frac{3}{x}y^{1/3} = \frac{12}{\sqrt{1+x^2}}. \tag{1.8.13}$$

We make the change of variables

$$u = y^{1/3}, \tag{1.8.14}$$

which implies that

$$\frac{du}{dx} = \frac{1}{3}y^{-2/3}\frac{dy}{dx}.$$

Substituting into Equation (1.8.13) yields

$$3\frac{du}{dx} + \frac{3}{x}u = \frac{12}{\sqrt{1+x^2}},$$

or in standard form,

$$\frac{du}{dx} + \frac{1}{x}u = \frac{4}{\sqrt{1+x^2}}. \tag{1.8.15}$$

An integrating factor for this linear equation is

$$I(x) = e^{\int(1/x)\,dx} = e^{\ln x} = x,$$

so that Equation (1.8.15) can be written as

$$\frac{d}{dx}(xu) = \frac{4x}{\sqrt{1+x^2}}.$$

Integrating, we obtain

$$u(x) = x^{-1}\left(4\sqrt{1+x^2} + c\right),$$

and so, from (1.8.14), the solution to the original differential equation is

$$y^{1/3} = x^{-1}\left(4\sqrt{1+x^2} + c\right). \qquad \square$$

Exercises for 1.8

Key Terms

Homogeneous of degree zero, Homogeneous first-order differential equation, Bernoulli equation.

Skills

- Be able to recognize whether or not a function $f(x, y)$ is homogeneous of degree zero, and whether or not a given differential equation is a homogeneous first-order differential equation.
- Know how to change the variables in a homogeneous first-order differential equation in order to get a differential equation that is separable and thus can be solved.
- Be able to recognize whether or not a given first-order differential equation is a Bernoulli equation.
- Know how to change the variables in a Bernoulli equation in order to get a differential equation that is first-order linear and thus can be solved.
- Be able to make other changes of variables to differential equations in order to turn them into differential equations that can be solved by methods from earlier in this chapter.

True-False Review

For Questions 1–9, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. The function

$$f(x, y) = \frac{2xy - x^2}{2xy + y^2}$$

is homogeneous of degree zero.

2. The function

$$f(x, y) = \frac{y^2}{x + y^2}$$

is homogeneous of degree zero.

3. The differential equation

$$\frac{dy}{dx} = \frac{1 + xy^2}{1 + x^2 y}$$

is a first-order homogeneous differential equation.

4. The differential equation

$$\frac{dy}{dx} = \frac{x^2 y^2}{x^4 + y^4}$$

is a first-order homogeneous differential equation.

5. The change of variables $y = xV(x)$ always turns a first-order homogeneous differential equation into a separable differential equation for V as a function of x.

6. The change of variables $u = y^{-n}$ always turns a Bernoulli differential equation into a first-order linear differential equation for u as a function of x.

7. The differential equation

$$\frac{dy}{dx} = \sqrt{x}y + \sqrt{xy}$$

is a Bernoulli differential equation.

8. The differential equation

$$\frac{dy}{dx} - e^{xy} y = 5x\sqrt{y}$$

is a Bernoulli differential equation.

9. The differential equation

$$\frac{dy}{dx} + xy = x^2 y^{2/3}$$

is a Bernoulli differential equation.

Problems

For Problems 1–8, determine whether the given function is homogeneous of degree zero. Rewrite those that are as functions of the single variable $V = y/x$.

1. $f(x, y) = \dfrac{x^2 - y^2}{xy}$.

2. $f(x, y) = x - y$.

3. $f(x, y) = \dfrac{x \sin(x/y) - y \cos(y/x)}{y}$.

4. $f(x, y) = \dfrac{\sqrt{x^2 + y^2}}{x - y}$, $\quad x > 0$.

5. $f(x, y) = \dfrac{y}{x-1}$.

6. $f(x, y) = \dfrac{x-3}{y} + \dfrac{5y+9}{3y}$.

7. $f(x, y) = \dfrac{\sqrt{x^2+y^2}}{x}, \quad x < 0$.

8. $f(x, y) = \dfrac{\sqrt{x^2+4y^2} - x + y}{x+3y}, \quad x, y \neq 0$.

For Problems 9–22, solve the given differential equation.

9. $(3x - 2y)\dfrac{dy}{dx} = 3y$.

10. $y' = \dfrac{(x+y)^2}{2x^2}$.

11. $\sin\left(\dfrac{y}{x}\right)(xy' - y) = x\cos\left(\dfrac{y}{x}\right)$.

12. $xy' = \sqrt{16x^2 - y^2} + y, \quad x > 0$.

13. $xy' - y = \sqrt{9x^2 + y^2}, \quad tx > 0$.

14. $y(x^2 - y^2)dx - x(x^2 + y^2)\,dy = 0$.

15. $xy' + y\ln x = y\ln y$.

16. $\dfrac{dy}{dx} = \dfrac{y^2 + 2xy - 2x^2}{x^2 - xy + y^2}$.

17. $2xy\,dy - (x^2 e^{-y^2/x^2} + 2y^2)\,dx = 0$.

18. $x^2\dfrac{dy}{dx} = y^2 + 3xy + x^2$.

19. $yy' = \sqrt{x^2 + y^2} - x, \quad x > 0$.

20. $2x(y + 2x)y' = y(4x - y)$.

21. $x\dfrac{dy}{dx} = x\tan(y/x) + y$.

22. $\dfrac{dy}{dx} = \dfrac{x\sqrt{x^2+y^2} + y^2}{xy}, \quad x > 0$.

23. Solve the differential equation in Example 1.8.6 by first transforming it into polar coordinates. [**Hint:** Write the differential equation in differential form and then express dx and dy in terms of r and θ.]

For Problems 24–26, solve the given initial-value problem.

24. $\dfrac{dy}{dx} = \dfrac{2(2y - x)}{x + y}, \quad y(1) = 2$.

25. $\dfrac{dy}{dx} = \dfrac{2x - y}{x + 4y}, \quad y(1) = 1$.

26. $\dfrac{dy}{dx} = \dfrac{y - \sqrt{x^2+y^2}}{x}, \quad y(3) = 4$.

27. Find *all* solutions to

$$x\frac{dy}{dx} - y = \sqrt{4x^2 - y^2}, \quad x > 0.$$

28. **(a)** Show that the general solution to the differential equation

$$\frac{dy}{dx} = \frac{x + ay}{ax - y}$$

can be written in polar form as $r = ke^{a\theta}$.

(b) For the particular case when $a = 1/2$, determine the solution satisfying the initial condition $y(1) = 1$, and find the maximum x-interval on which this solution is valid. [**Hint:** When does the solution curve have a vertical tangent?]

(c) ◇ On the same set of axes, sketch the spiral corresponding to your solution in (b), and the line $y = x/2$. Thus verify the x-interval obtained in (b) with the graph.

For Problems 29–30, determine the orthogonal trajectories to the given family of curves. Sketch some curves from each family.

29. $x^2 + y^2 = 2cy$.

30. $(x - c)^2 + (y - c)^2 = 2c^2$.

31. Fix a real number m. Let S_1 denote the family of circles, centered on the line $y = mx$, each member of which passes through the origin.

(a) Show that the equation of S_1 can be written in the form

$$(x - a)^2 + (y - ma)^2 = a^2(m^2 + 1),$$

where a is a constant that labels particular members of the family.

(b) Determine the equation of the family of orthogonal trajectories to S_1, and show that it consists of the family of circles centered on the line $x = -my$ that pass through the origin.

(c) ◇ Sketch some curves from both families when $m = \sqrt{3}/3$.

Let F_1 and F_2 be two families of curves with the property that whenever a curve from the family F_1 intersects one from the family F_2, it does so at an angle $\alpha \neq \pi/2$. If we know the equation of F_2, then it can be shown (see Problem 26 in Section 1.1) that the differential equation for determining F_1 is

$$\frac{dy}{dx} = \frac{m_2 - \tan\alpha}{1 + m_2 \tan\alpha}, \qquad (1.8.16)$$

where m_2 denotes the slope of the family F_2 at the point (x, y).

For Problems 32–34, use Equation (1.8.16) to determine the equation of the family of curves that cuts the given family at an angle $\alpha = \pi/4$.

32. $x^2 + y^2 = c$.

33. $y = cx^6$.

34. $x^2 + y^2 = 2cx$.

35. **(a)** Use Equation (1.8.16) to find the equation of the family of curves that intersects the family of hyperbolas $y = c/x$ at an angle $\alpha = \alpha_0$.

(b) ◇ When $\alpha_0 = \pi/4$, sketch several curves from each family.

36. **(a)** Use Equation (1.8.16) to show that the family of curves that intersects the family of concentric circles $x^2 + y^2 = c$ at an angle $\alpha = \tan^{-1} m$ has polar equation $r = ke^{m\theta}$.

(b) ◇ When $\alpha = \pi/6$, sketch several curves from each family.

For Problems 37–49, solve the given differential equation.

37. $y' - x^{-1}y = 4x^2y^{-1}\cos x, \quad x > 0.$

38. $\dfrac{dy}{dx} + \dfrac{1}{2}(\tan x)y = 2y^3 \sin x.$

39. $\dfrac{dy}{dx} - \dfrac{3}{2x}y = 6y^{1/3}x^2 \ln x.$

40. $y' + 2x^{-1}y = 6\sqrt{1 + x^2}\sqrt{y}, \quad x > 0.$

41. $y' + 2x^{-1}y = 6y^2x^4.$

42. $2x(y' + y^3x^2) + y = 0.$

43. $(x - a)(x - b)(y' - \sqrt{y}) = 2(b - a)y$, where a, b are constants.

44. $y' + 6x^{-1}y = 3x^{-1}y^{2/3}\cos x, \quad x > 0.$

45. $y' + 4xy = 4x^3y^{1/2}.$

46. $\dfrac{dy}{dx} - \dfrac{1}{2x \ln x}y = 2xy^3.$

47. $\dfrac{dy}{dx} - \dfrac{1}{(\pi - 1)x}y = \dfrac{3}{1 - \pi}xy^{\pi}.$

48. $2y' + y\cot x = 8y^{-1}\cos^3 x.$

49. $(1 - \sqrt{3})y' + y\sec x = y^{\sqrt{3}}\sec x.$

For Problems 50–51, solve the given initial-value problem.

50. $\dfrac{dy}{dx} + \dfrac{2x}{1 + x^2}y = xy^2, \quad y(0) = 1.$

51. $y' + y\cot x = y^3\sin^3 x, \quad y(\pi/2) = 1.$

52. Consider the differential equation

$$y' = F(ax + by + c), \qquad (1.8.17)$$

where $a, b \neq 0$, and c are constants. Show that the change of variables from x, y to x, V, where

$$V = ax + by + c$$

reduces Equation (1.8.17) to the separable form

$$\frac{1}{bF(V) + a}\,dV = dx.$$

For Problems 53–55, use the result from the previous problem to solve the given differential equation. For Problem 53, impose the given initial condition as well.

53. $y' = (9x - y)^2, \quad y(0) = 0.$

54. $y' = (4x + y + 2)^2.$

55. $y' = \sin^2(3x - 3y + 1).$

56. Show that the change of variables $V = xy$ transforms the differential equation

$$\frac{dy}{dx} = \frac{y}{x}F(xy)$$

into the separable differential equation

$$\frac{1}{V[F(V) + 1]}\frac{dV}{dx} = \frac{1}{x}.$$

57. Use the result from the previous problem to solve

$$\frac{dy}{dx} = \frac{y}{x}[\ln(xy) - 1].$$

58. Consider the differential equation

$$\frac{dy}{dx} = \frac{x+2y-1}{2x-y+3}. \tag{1.8.18}$$

(a) Show that the change of variables defined by

$$x = u - 1, \quad y = v + 1$$

transforms Equation (1.8.18) into the homogeneous equation

$$\frac{dv}{du} = \frac{u+2v}{2u-v}. \tag{1.8.19}$$

(b) Find the general solution to Equation (1.8.19), and hence solve Equation (1.8.18).

59. A differential equation of the form

$$y' + p(x)y + q(x)y^2 = r(x) \tag{1.8.20}$$

is called a **Riccati** equation.

(a) If $y = Y(x)$ is a known solution to Equation (1.8.20), show that the substitution

$$y = Y(x) + v^{-1}(x)$$

reduces it to the linear equation

$$v' - [p(x) + 2Y(x)q(x)]v = q(x).$$

(b) Find the general solution to the Riccati equation

$$x^2y' - xy - x^2y^2 = 1, \quad x > 0,$$

given that $y = -x^{-1}$ is a solution.

60. Consider the Riccati equation

$$y' + 2x^{-1}y - y^2 = -2x^{-2}, \quad x > 0. \tag{1.8.21}$$

(a) Determine the values of the constants a and r such that $y(x) = ax^r$ is a solution to Equation (1.8.21).

(b) Use the result from part (a) of the previous problem to determine the general solution to Equation (1.8.21).

61. **(a)** Show that the change of variables $y = x^{-1} + w$ transforms the Riccati differential equation

$$y' + 7x^{-1}y - 3y^2 = 3x^{-2} \tag{1.8.22}$$

into the Bernoulli equation

$$w' + x^{-1}w = 3w^2. \tag{1.8.23}$$

(b) Solve Equation (1.8.23), and hence determine the general solution to (1.8.22).

62. Consider the differential equation

$$y^{-1}y' + p(x)\ln y = q(x), \tag{1.8.24}$$

where $p(x)$ and $q(x)$ are continuous functions on some interval (a, b). Show that the change of variables $u = \ln y$ reduces Equation (1.8.24) to the linear differential equation

$$u' + p(x)u = q(x),$$

and hence show that the general solution to Equation (1.8.24) is

$$y(x) = \exp\left\{I^{-1}\left[\int I(x)q(x)\,dx + c\right]\right\},$$

where

$$I = e^{\int p(x)dx} \tag{1.8.25}$$

and c is an arbitrary constant.

63. Use the technique derived in the previous problem to solve the initial-value problem

$$y^{-1}y' - 2x^{-1}\ln y = x^{-1}(1 - 2\ln x),$$

$$y(1) = e.$$

64. Consider the differential equation

$$f'(y)\frac{dy}{dx} + p(x)f(y) = q(x), \tag{1.8.26}$$

where p and q are continuous functions on some interval (a, b), and f is an invertible function. Show that Equation (1.8.26) can be written as

$$\frac{du}{dx} + p(x)u = q(x),$$

where $u = f(y)$, and hence show that the general solution to Equation (1.8.26) is

$$y(x) = f^{-1}\left\{I^{-1}\left[\int I(x)q(x)\,dx + c\right]\right\},$$

where I is given in (1.8.25), f^{-1} is the inverse of f, and c is an arbitrary constant.

65. Solve

$$\sec^2 y\,\frac{dy}{dx} + \frac{1}{2\sqrt{1+x}}\tan y = \frac{1}{2\sqrt{1+x}}.$$

1.9 Exact Differential Equations

For the next technique it is best to consider first-order differential equations written in differential form

$$M(x, y)\,dx + N(x, y)\,dy = 0, \tag{1.9.1}$$

where M and N are given functions, assumed to be sufficiently smooth.[8] The method that we will consider is based on the idea of a differential. Recall from a previous calculus course that if $\phi = \phi(x, y)$ is a function of two variables, x and y, then the differential of ϕ, denoted $d\phi$, is defined by

$$d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy. \tag{1.9.2}$$

Example 1.9.1 Solve

$$2x \sin y\,dx + x^2 \cos y\,dy = 0. \tag{1.9.3}$$

Solution: This equation is separable, but we will use a different technique to solve it. By inspection, we notice that

$$2x \sin y\,dx + x^2 \cos y\,dy = d(x^2 \sin y).$$

Consequently, Equation (1.9.3) can be written as $d(x^2 \sin y) = 0$, which implies that $x^2 \sin y$ is constant, hence the general solution to Equation (1.9.3) is

$$\sin y = \frac{c}{x^2},$$

where c is an arbitrary constant. □

In the foregoing example we were able to write the given differential equation in the form $d\phi(x, y) = 0$, and hence obtain its solution. However, we cannot always do this. Indeed we see by comparing Equation (1.9.1) with (1.9.2) that the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

can be written as $d\phi = 0$ if and only if

$$M = \frac{\partial \phi}{\partial x} \quad \text{and} \quad N = \frac{\partial \phi}{\partial y}$$

for some function ϕ. This motivates the following definition:

[8] This means we assume that the functions M and N have continuous derivatives of sufficiently high order.