25. Construct *distinct* matrix functions A and B defined on all of $\mathbb{R}$ such that $A(0) = B(0)$ and $A(1) = B(1)$.

26. Prove that a symmetric upper triangular matrix is diagonal.

27. Determine all elements of the 3×3 skew-symmetric matrix A with $a_{21} = 1, a_{31} = 3, a_{23} = -1$.

2.2 Matrix Algebra

In the previous section we introduced the general idea of a matrix. The next step is to develop the algebra of matrices. Unless otherwise stated, we assume that all elements of the matrices that appear are real or complex numbers.

Addition and Subtraction of Matrices and Multiplication of a Matrix by a Scalar

Addition and subtraction of matrices is defined only for matrices with the same dimensions. We begin with addition.

DEFINITION 2.2.1

If A and B are both $m \times n$ matrices, then we define **addition** (or the **sum**) of A and B, denoted by $A + B$, to be the $m \times n$ matrix whose elements are obtained by adding corresponding elements of A and B. In index notation, if $A = [a_{ij}]$ and $B = [b_{ij}]$, then $A + B = [a_{ij} + b_{ij}]$.

Example 2.2.2 We have

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & -5 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 5 \\ -5 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 8 \\ -1 & -3 & 7 \end{bmatrix}. \qquad \Box$$

Properties of Matrix Addition: If A and B are both $m \times n$ matrices, then

$$A + B = B + A \qquad \text{(matrix addition is commutative)},$$
$$A + (B + C) = (A + B) + C \qquad \text{(matrix addition is associative)}.$$

Both of these properties follow directly from Definition 2.2.1.

In order that we can model oscillatory physical phenomena, in much of the later work we will need to use complex as well as real numbers. Throughout the text we will use the term **scalar** to mean a real or complex number.

DEFINITION 2.2.3

If A is an $m \times n$ matrix and s is a scalar, then we let sA denote the matrix obtained by multiplying every element of A by s. This procedure is called **scalar multiplication**. In index notation, if $A = [a_{ij}]$, then $sA = [sa_{ij}]$.

Example 2.2.4 If $A = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix}$, then $5A = \begin{bmatrix} 10 & -5 \\ 20 & 30 \end{bmatrix}$. $\Box$

Example 2.2.5 If $A = \begin{bmatrix} 1+i & i \\ 2+3i & 4 \end{bmatrix}$ and $s = 1 - 2i$, where $i = \sqrt{-1}$, find sA.

Solution: We have

$$sA = \begin{bmatrix} (1-2i)(1+i) & (1-2i)i \\ (1-2i)(2+3i) & (1-2i)4 \end{bmatrix} = \begin{bmatrix} 3-i & 2+i \\ 8-i & 4-8i \end{bmatrix}. \qquad \square$$

DEFINITION 2.2.6

We define **subtraction** of two matrices with the *same dimensions* by

$$A - B = A + (-1)B.$$

In index notation, $A - B = [a_{ij} - b_{ij}]$. That is, we subtract corresponding elements.

Further properties satisfied by the operations of matrix addition and multiplication of a matrix by a scalar are as follows:

Properties of Scalar Multiplication: For any scalars s and t, and for any matrices A and B of the same size,

$$\begin{aligned} 1A &= A && \text{(unit property)}, \\ s(A+B) &= sA + sB && \text{(distributivity of scalars over matrix addition)}, \\ (s+t)A &= sA + tA && \text{(distributivity of scalar addition over matrices)}, \\ s(tA) = (st)A &= (ts)A = t(sA) && \text{(associativity of scalar multiplication)}. \end{aligned}$$

The $m \times n$ **zero matrix**, denoted $0_{m \times n}$ (or simply 0, if the dimensions are clear), is the $m \times n$ matrix whose elements are all zeros. In the case of the $n \times n$ zero matrix, we may write 0_n. We now collect a few properties of the zero matrix. The first of these below indicates that the zero matrix plays a similar role in matrix addition to that played by the number zero in the addition of real numbers.

Properties of the Zero Matrix: For all matrices A and the zero matrix of the same size, we have

$$A + 0 = A, \qquad A - A = 0, \qquad \text{and} \qquad 0A = 0.$$

Note that in the last property here, the zero on the left side of the equation is a scalar, while the zero on the right side of the equation is a matrix.

Multiplication of Matrices

The definition we introduced above for how to multiply a matrix by a scalar is essentially the only possibility if, in the case when s is a positive integer, we want sA to be the same matrix as the one obtained when A is added to itself s times. We now define how to multiply two matrices together. In this case the multiplication operation is by no means obvious. However, in Chapter 5 when we study linear transformations, the motivation for the matrix multiplication procedure we are defining here will become quite transparent (see Theorem 5.5.7).

We will build up to the general definition of matrix multiplication in three stages.

Case 1: Product of a row n-vector and a column n-vector. We begin by generalizing a concept from elementary calculus. If **a** and **b** are either row or column n-vectors, with components $a_1, a_2, \ldots, a_n$, and $b_1, b_2, \ldots, b_n$, respectively, then their **dot product**, denoted $\mathbf{a} \cdot \mathbf{b}$, is the *number*

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

As we will see, this is the key formula in defining the product of two matrices. Now let $\mathbf{a}$ be a *row* n-vector, and let $\mathbf{x}$ be a *column* n-vector. Then their **matrix product ax** is defined to be the 1×1 matrix whose single element is obtained by taking the dot product of the row vectors $\mathbf{a}$ and $\mathbf{x}^T$. Thus,

$$\mathbf{ax} = \begin{bmatrix} a_1 & a_1 & \ldots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [a_1 x_1 + a_2 x_2 + \cdots + a_n x_n].$$

Example 2.2.7 If $\mathbf{a} = \begin{bmatrix} 2 & -1 & 3 & 5 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ -3 \\ 4 \end{bmatrix}$, then

$$\mathbf{ax} = \begin{bmatrix} 2 & -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -3 \\ 4 \end{bmatrix} = [(2)(3) + (-1)(2) + (3)(-3) + (5)(4)] = [15]. \quad \square$$

Case 2: Product of an $m \times n$ matrix and a column n-vector. If A is an $m \times n$ matrix and $\mathbf{x}$ is a column n-vector, then the product $A\mathbf{x}$ is defined to be the $m \times 1$ matrix whose ith element is obtained by taking the dot product of the ith row vector of A with $\mathbf{x}$. (See Figure 2.2.1.)

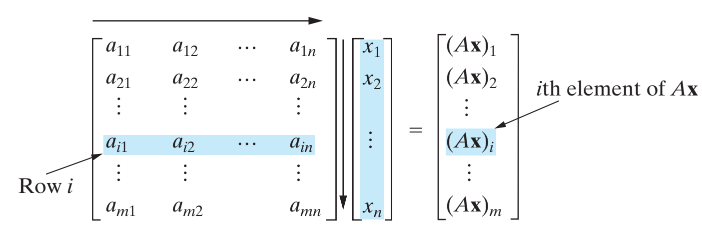

Figure 2.2.1: Multiplication of an $m \times n$ matrix with a column n-vector.

The ith row vector of A, $\mathbf{a}_i$, is

$$\mathbf{a}_i = \begin{bmatrix} a_{i1} & a_{i2} & \ldots & a_{in} \end{bmatrix},$$

so that $A\mathbf{x}$ has ith element

$$(A\mathbf{x})_i = a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n.$$

Consequently the column vector $A\mathbf{x}$ has elements

$$(A\mathbf{x})_i = \sum_{k=1}^{n} a_{ik} x_k, \qquad 1 \leq i \leq m. \tag{2.2.1}$$

As illustrated in the next example, in practice, we do not use the formula (2.2.1); rather, we explicitly take the matrix products of the row vectors of A with the column vector $\mathbf{x}$.

Example 2.2.8 Find $A\mathbf{x}$ if $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 4 & -6 \\ 5 & -2 & 0 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix}$.

Solution: We have

$$A\mathbf{x} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 4 & -6 \\ 5 & -2 & 0 \end{bmatrix} \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ 41 \end{bmatrix}.$$

□

The following result regarding multiplication of a column vector by a matrix will be used repeatedly in later chapters.

Theorem 2.2.9 If $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$ is an $m \times n$ matrix and $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is a column n-vector, then

$$A\mathbf{c} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n. \tag{2.2.2}$$

Proof The element a_{ik} of A is the ith component of the column m-vector $\mathbf{a}_k$, so

$$a_{ik} = (\mathbf{a}_k)_i.$$

Applying formula (2.2.1) for multiplication of a column vector by a matrix yields

$$(A\mathbf{c})_i = \sum_{k=1}^{n} a_{ik}c_k = \sum_{k=1}^{n} (\mathbf{a}_k)_i c_k = \sum_{k=1}^{n} (c_k\mathbf{a}_k)_i.$$

Consequently,

$$A\mathbf{c} = \sum_{k=1}^{n} c_k\mathbf{a}_k = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n$$

as required. ■

If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are column m-vectors and $c_1, c_2, \ldots, c_n$ are scalars, then an expression of the form

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

is called a **linear combination** of the column vectors. Therefore, from Equation (2.2.2), we see that the vector $A\mathbf{c}$ is obtained by taking a linear combination of the column vectors of A. For example, if

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 5 \\ -1 \end{bmatrix},$$

then

$$A\mathbf{c} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = 5\begin{bmatrix} 2 \\ 4 \end{bmatrix} + (-1)\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 17 \end{bmatrix}.$$

Case 3: Product of an $m \times n$ matrix and an $n \times p$ matrix. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product AB has columns defined by multiplying the matrix A by the respective column vectors of B, as described in Case 2. That is, if $B = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p]$, then AB is the $m \times p$ matrix defined by

$$AB = [A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_p].$$

Example 2.2.10 If $A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 5 & -2 \\ 8 & 4 \end{bmatrix}$, determine AB.

Solution: We have

$$AB = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 5 & 7 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 5 & -2 \\ 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} [(1)(2) + (4)(5) + (2)(8)] & [(1)(3) + (4)(-2) + (2)(4)] \\ [(3)(2) + (5)(5) + (7)(8)] & [(3)(3) + (5)(-2) + (7)(4)] \end{bmatrix} = \begin{bmatrix} 38 & 3 \\ 87 & 27 \end{bmatrix}. \quad \square$$

Example 2.2.11 If $A = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \end{bmatrix}$, determine AB.

Solution: We have

$$AB = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\begin{bmatrix} 2 & 4 \end{bmatrix} = \begin{bmatrix} (2)(2) & (2)(4) \\ (-1)(2) & (-1)(4) \\ (3)(2) & (3)(4) \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ -2 & -4 \\ 6 & 12 \end{bmatrix}. \quad \square$$

Another way to describe AB is to note that the element $(AB)_{ij}$ is obtained by computing the matrix product of the ith row vector of A and the jth column vector of B. That is,

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

Expressing this using the summation notation yields the following result:

DEFINITION 2.2.12

If $A = [a_{ij}]$ is an $m \times n$ matrix, $B = [b_{ij}]$ is an $n \times p$ matrix, and $C = AB$, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}, \qquad 1 \le i \le m, \quad 1 \le j \le p. \tag{2.2.3}$$

This is called the **index form** of the matrix product.

The formula (2.2.3) for the ijth element of AB is very important and will often be required in the future. The reader should memorize it.

In order for the product AB to be defined, we see that A and B must satisfy

$$\text{number of columns of } A = \text{number of rows of } B.$$

In such a case, if C represents the product matrix AB, then the relationship between the dimensions of the matrices is

$$\boxed{A_{m\times n} \quad B_{n\times p} = C_{m\times p}}$$

SAME

RESULT

Now we give some further examples of matrix multiplication.

Example 2.2.13 If $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 5 & 3 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ 1 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 13 & 9 \\ 8 & 16 & 12 \end{bmatrix}.$$ □

Example 2.2.14 If $A = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \end{bmatrix}.$$ □

Example 2.2.15 If $A = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 & -6 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & -6 \end{bmatrix} = \begin{bmatrix} 2 & 8 & -12 \\ -1 & -4 & 6 \\ 3 & 12 & -18 \end{bmatrix}.$$ □

Example 2.2.16 If $A = \begin{bmatrix} 1-i & i \\ 2+i & 1+i \end{bmatrix}$ and $B = \begin{bmatrix} 3+2i & 1+4i \\ i & -1+2i \end{bmatrix}$, then

$$AB = \begin{bmatrix} 1-i & i \\ 2+i & 1+i \end{bmatrix} \begin{bmatrix} 3+2i & 1+4i \\ i & -1+2i \end{bmatrix} = \begin{bmatrix} 4-i & 3+2i \\ 3+8i & -5+10i \end{bmatrix}.$$ □

Notice that in Examples 2.2.13 and 2.2.14 above, the product BA is not defined, since the number of columns of the matrix B does not agree with the number of rows of the matrix A.

We can now establish some basic properties of matrix multiplication.

Theorem 2.2.17 If A, B and C have appropriate dimensions for the operations to be performed, then

$$A(BC) = (AB)C \qquad \text{(associativity of matrix multiplication)}, \qquad (2.2.4)$$

$$A(B+C) = AB + AC \qquad \text{(left distributivity of matrix multiplication)}, \qquad (2.2.5)$$

$$(A+B)C = AC + BC \qquad \text{(right distributivity of matrix multiplication)}. \qquad (2.2.6)$$

Proof The idea behind the proof of each of these results is to use the definition of matrix multiplication to show that the ijth element of the matrix on the left-hand side of each equation is equal to the ijth element of the matrix on the right-hand side. We illustrate by proving (2.2.6), but we leave the proofs of (2.2.4) and (2.2.5) as exercises. Suppose that A and B are $m \times n$ matrices and that C is an $n \times p$ matrix. Then, from Equation (2.2.3),

$$\begin{aligned}
[(A+B)C]_{ij} &= \sum_{k=1}^{n}(a_{ik}+b_{ik})c_{kj} = \sum_{k=1}^{n} a_{ik}c_{kj} + \sum_{k=1}^{n} b_{ik}c_{kj} \\
&= (AC)_{ij} + (BC)_{ij} \\
&= (AC+BC)_{ij}, \qquad 1 \le i \le m, \quad 1 \le j \le p.
\end{aligned}$$

Consequently,

$$(A+B)C = AC + BC. \qquad \blacksquare$$

Theorem 2.2.17 states that matrix multiplication is associative and distributive (over addition). We now consider the question of commutativity of matrix multiplication. If A is an $m \times n$ matrix and B is an $n \times m$ matrix, we can form both of the products AB and BA, which are $m \times m$ and $n \times n$, respectively. In the first of these, we say that B has been **premultiplied** by A, whereas in the second, we say that B has been **postmultiplied** by A. If $m \neq n$, then the matrices AB and BA will have different dimensions, so they cannot be equal. It is important to realize, however, that even if $m = n$, in general (that is, except for special cases)

$$AB \neq BA.$$

This is the statement that

matrix multiplication is not commutative.

With a little bit of thought this should not be too surprising, in view of the fact that the ijth element of AB is obtained by taking the matrix product of the ith row vector of A with the jth column vector of B, whereas the ijth element of BA is obtained by taking the matrix product of the ith row vector of B with the jth column vector of A. We illustrate with an example.

Example 2.2.18 If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$, find AB and BA.

Solution: We have

$$AB = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 3 & -4 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 9 \\ 3 & 1 \end{bmatrix}.$$

Thus we see that in this example, $AB \neq BA$. $\square$

As an exercise, the reader can calculate the matrix BA in Examples 2.2.15 and 2.2.16 and again see that $AB \neq BA$.

For an $n \times n$ matrix we use the usual power notation to denote the operation of multiplying A by itself. Thus,

$$A^2 = AA, \qquad A^3 = AAA, \qquad \text{and so on.}$$

The **identity matrix**, I_n (or just I if the dimensions are obvious), is the $n \times n$ matrix with ones on the main diagonal and zeros elsewhere. For example,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

DEFINITION 2.2.19

The elements of I_n can be represented by the **Kronecker delta symbol**, δ_{ij}, defined by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Then,

$$I_n = [\delta_{ij}].$$

The following properties of the identity matrix indicate that it plays the same role in matrix multiplication as the number 1 does in the multiplication of real numbers.

Properties of the Identity Matrix:

1. $A_{m\times n} I_n = A_{m\times n}$.
2. $I_m A_{m\times p} = A_{m\times p}$.

Proof We establish property 1 and leave the proof of property 2 as an exercise (Problem 25). Using the index form of the matrix product, we have

$$(AI)_{ij} = \sum_{k=1}^{n} a_{ik}\delta_{kj} = a_{i1}\delta_{1j} + a_{i2}\delta_{2j} + \cdots + a_{ij}\delta_{jj} + \cdots + a_{in}\delta_{nj}.$$

But, from the definition of the Kronecker delta symbol, we see that all terms in the summation with $k \neq j$ vanish, so that we are left with

$$(AI)_{ij} = a_{ij}\delta_{jj} = a_{ij}, \qquad 1 \leq i \leq m, \quad 1 \leq j \leq n. \qquad \blacksquare$$

The next example illustrates property 2 of the identity matrix.

Example 2.2.20 If $A = \begin{bmatrix} 2 & -1 \\ 3 & 5 \\ 0 & -2 \end{bmatrix}$, verify that $I_3 A = A$.

Solution: We have

$$I_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 5 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 5 \\ 0 & -2 \end{bmatrix} = A. \qquad \square$$

Properties of the Transpose

The operation of taking the transpose of a matrix was introduced in the previous section. The next theorem gives three important properties satisfied by the transpose. These should be memorized.

Theorem 2.2.21 Let A and C be $m \times n$ matrices, and let B be an $n \times p$ matrix. Then

1. $(A^T)^T = A$.
2. $(A + C)^T = A^T + C^T$.
3. $(AB)^T = B^T A^T$.

Proof For all three statements, our strategy is again to show that the (i, j)-elements of each side of the equation are the same. We prove statement 3 and leave the proofs of 1 and 2 for the exercises (Problem 24). From the definition of the transpose and the index form of the matrix product, we have

$$\begin{aligned}
[(AB)^T]_{ij} &= (AB)_{ji} && \text{(definition of the transpose)} \\
&= \sum_{k=1}^{n} a_{jk} b_{ki} && \text{(index form of the matrix product)} \\
&= \sum_{k=1}^{n} b_{ki} a_{jk} = \sum_{k=1}^{n} b^T_{ik} a^T_{kj} \\
&= (B^T A^T)_{ij}.
\end{aligned}$$

Consequently,

$$(AB)^T = B^T A^T. \qquad \blacksquare$$

Results for Triangular Matrices

Upper and lower triangular matrices play a significant role in the analysis of linear systems of equations. The following theorem and its corollary will be needed in Section 2.7.

Theorem 2.2.22 The product of two lower (upper) triangular matrices is a lower (upper) triangular matrix.

Proof Suppose that A and B are $n \times n$ lower triangular matrices. Then, $a_{ik} = 0$ whenever $i < k$, and $b_{kj} = 0$ whenever $k < j$. If we let $C = AB$, then we must prove that

$$c_{ij} = 0 \quad \text{whenever} \quad i < j.$$

Using the index form of the matrix product, we have

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=j}^{n} a_{ik} b_{kj} \qquad (\text{since } b_{kj} = 0 \text{ if } k < j). \tag{2.2.7}$$

We now impose the condition that $i < j$. Then, since $k \geq j$ in (2.2.7), it follows that $k > i$. However, this implies that $a_{ik} = 0$ (since A is lower triangular), and hence, from (2.2.7), that

$$c_{ij} = 0 \quad \text{whenever} \quad i < j.$$

as required.

To establish the result for upper triangular matrices, either we can give an argument similar to that presented above for lower triangular matrices, or we can use the fact that the transpose of a lower triangular matrix is an upper triangular matrix, and vice versa. Hence, if A and B are $n \times n$ upper triangular matrices, then A^T and B^T are lower triangular, and therefore by what we proved above, $(AB)^T = B^T A^T$ remains lower triangular. Thus, AB is upper triangular. ■

Corollary 2.2.23 The product of two unit lower (upper) triangular matrices is a unit lower (upper) triangular matrix.

Proof Let A and B be unit lower triangular $n \times n$ matrices. We know from Theorem 2.2.22 that $C = AB$ is a lower triangular matrix. We must establish that $c_{ii} = 1$ for each i. The elements on the main diagonal of C can be obtained by setting $j = i$ in (2.2.7):

$$c_{ii} = \sum_{k=i}^{n} a_{ik} b_{ki}. \tag{2.2.8}$$

Since $a_{ik} = 0$ whenever $k > i$, the only nonzero term in the summation in (2.2.8) occurs when $k = i$. Consequently,

$$c_{ii} = a_{ii} b_{ii} = 1 \cdot 1 = 1, \qquad i = 1, 2, \dots, n.$$

The proof for unit upper triangular matrices is similar and left as an exercise. ■

The Algebra and Calculus of Matrix Functions

By and large, the algebra of matrix and vector functions is the same as that for matrices and vectors of real or complex numbers. Since vector functions are a special case of matrix functions, we focus here on matrix functions. The main comment here pertains to scalar multiplication. In the description of scalar multiplication of matrices of numbers, the scalars were required to be real or complex numbers. However, for matrix functions, we can scalar multiply by any *scalar function* $s(t)$.

Example 2.2.24 If $s(t) = e^t$ and $A(t) = \begin{bmatrix} -2+t & e^{2t} \\ 4 & \cos t \end{bmatrix}$, then

$$s(t)A(t) = \begin{bmatrix} e^t(-2+t) & e^{3t} \\ 4e^t & e^t \cos t \end{bmatrix}.$$ □

Example 2.2.25 Referring to A and B from Example 2.1.14, find $2A - tB^T$.

Solution: We have

$$2A - tB^T = \begin{bmatrix} 2t^3 & 2t - 2\cos t & 10 \\ 2e^{t^2} & 2\ln(t+1) & 2te^t \end{bmatrix} - \begin{bmatrix} 5t - t^2 + t^3 & -t & 6t \\ t\sin(e^{2t}) & t\tan t & 6t - t^2 \end{bmatrix}$$
$$= \begin{bmatrix} t^3 + t^2 - 5t & 3t - 2\cos t & 10 - 6t \\ 2e^{t^2} - t\sin(e^{2t}) & 2\ln(t+1) - t\tan t & 2te^t + t^2 - 6t \end{bmatrix}. \quad \square$$

We can also perform calculus operations on matrix functions. In particular we can differentiate and integrate them. The rules for doing so are as follows:

1. The derivative of a matrix function is obtained by differentiating *every* element of the matrix. Thus, if $A(t) = [a_{ij}(t)]$, then

$$\frac{dA}{dt} = \left[\frac{da_{ij}(t)}{dt}\right],$$

provided that each of the a_{ij} is differentiable.

2. It follows from (1) and the index form of the matrix product that if A and B are both differentiable and the product AB is defined, then

$$\frac{d}{dt}(AB) = A\frac{dB}{dt} + \frac{dA}{dt}B.$$

The key point to notice is that the order of the multiplication must be preserved.

3. If $A(t) = [a_{ij}(t)]$, where each $a_{ij}(t)$ is integrable on an interval $[a, b]$, then

$$\int_a^b A(t)\,dt = \left[\int_a^b a_{ij}(t)\,dt\right].$$

Example 2.2.26 If $A(t) = \begin{bmatrix} 2t & 1 \\ 6t^2 & 4e^{2t} \end{bmatrix}$, determine dA/dt and $\int_0^1 A(t)\,dt$.

Solution: We have

$$\frac{dA}{dt} = \begin{bmatrix} 2 & 0 \\ 12t & 8e^{2t} \end{bmatrix},$$

whereas

$$\int_0^1 A(t)\,dt = \begin{bmatrix} \int_0^1 2t\,dt & \int_0^1 1\,dt \\ \int_0^1 6t^2\,dt & \int_0^1 4e^{2t}\,dt \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2(e^2 - 1) \end{bmatrix}. \quad \square$$

Exercises for 2.2

Key Terms

Matrix addition and subtraction, Scalar multiplication, Matrix multiplication, Dot product, Linear combination of column vectors, Index form, Premultiplication, Postmultiplication, Zero matrix, Identity matrix, Kronecker delta symbol.

Skills

- Be able to perform matrix addition, subtraction, and multiplication.
- Know the basic relationships between the dimensions of two matrices A and B in order for $A + B$ to be defined, and in order for AB to be defined.
- Be able to multiply a matrix by a scalar.
- Be able to express the product $A\mathbf{x}$ of a matrix and a column vector as a linear combination of the columns of A.

- Be familiar with all of the basic properties of matrix addition, matrix multiplication, scalar multiplication, the zero matrix, the identity matrix, the transpose of a matrix, and lower (upper) triangular matrices.
- Know the basic technique for showing formally that two matrices are equal.
- Be able to perform algebra and calculus operations on matrix functions.

True-False Review

For Questions 1–12, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. For all matrices A, B, and C of the appropriate dimensions, we have

$$(AB)C = (CA)B.$$

2. If A is an $m \times n$ matrix, B is an $n \times p$ matrix, and C is a $p \times q$ matrix, then ABC is an $m \times q$ matrix.

3. If A and B are symmetric $n \times n$ matrices, then so is $A + B$.

4. If A and B are skew-symmetric $n \times n$ matrices, then AB is a symmetric matrix.

5. For $n \times n$ matrices A and B, we have

$$(A + B)^2 = A^2 + 2AB + B^2.$$

6. If $AB = 0$, then either $A = 0$ or $B = 0$.

7. If A and B are square matrices such that AB is upper triangular, then A and B must both be upper triangular.

8. If A is a square matrix such that $A^2 = A$, then A must be the zero matrix or the identity matrix.

9. If A is a matrix of numbers, then if we consider A as a matrix function, its derivative is the zero matrix.

10. If A and B are matrix functions whose product AB is defined, then

$$\frac{d}{dt}(AB) = A\frac{dB}{dt} + B\frac{dA}{dt}.$$

11. If A is an $n \times n$ matrix function such that A and dA/dt are the same function, then $A = ce^t I_n$ for some constant c.

12. If A and B are matrix functions whose product AB is defined, then the matrix functions $(AB)^T$ and $B^T A^T$ are the same.

Problems

1. If

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 5 \end{bmatrix},$$

find $2A$, $-3B$, $A - 2B$, and $3A + 4B$.

2. If

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{bmatrix},$$

find the matrix D such that $2A + B - 3C + 2D = A + 4C$.

3. Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 1 & 2 \\ 4 & 6 & -2 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & -2 & 3 \end{bmatrix}.$$

Find, if possible, AB, BC, CA, DC, DB, AD, and CD.

For Problems 4–6, determine AB for the given matrices. In these problems i denotes $\sqrt{-1}$.

4. $A = \begin{bmatrix} 2-i & 1+i \\ -i & 2+4i \end{bmatrix}$, $\quad B = \begin{bmatrix} i & 1-3i \\ 0 & 4+i \end{bmatrix}$.

5. $A = \begin{bmatrix} 3+2i & 2-4i \\ 5+i & -1+3i \end{bmatrix}$, $\quad B = \begin{bmatrix} -1+i & 3+2i \\ 4-3i & 1+i \end{bmatrix}$.

6. $A = \begin{bmatrix} 3-2i & i \\ -i & 1 \end{bmatrix}$, $\quad B = \begin{bmatrix} -1+i & 2-i & 0 \\ 1+5i & 0 & 3-2i \end{bmatrix}$.

7. Let

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ -2 & 3 & 4 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 2 \\ 1 & 5 \\ 4 & -3 \\ -1 & 6 \end{bmatrix},$$

$$C = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix}.$$

Find ABC and CAB.

8. If

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 2 \\ 5 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 \\ -1 \end{bmatrix},$$

find $(2A - 3B)C$.

For Problems 9–11, determine $A\mathbf{c}$ by computing an appropriate linear combination of the column vectors of A.

9. $A = \begin{bmatrix} 1 & 3 \\ -5 & 4 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$.

10. $A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 1 & 5 \\ 7 & -6 & 3 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$.

11. $A = \begin{bmatrix} -1 & 2 \\ 4 & 7 \\ 5 & -4 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

12. If A is an $m \times n$ matrix and C is an $r \times s$ matrix, what must be the dimensions of B in order for the product ABC to be defined? Write an expression for the (i, j)th element of ABC in terms of the elements of A, B and C.

13. Find A^2, A^3, and A^4 if

(a) $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix}$.

14. If A and B are $n \times n$ matrices, prove that

(a) $(A + B)^2 = A^2 + AB + BA + B^2$.

(b) $(A - B)^2 = A^2 - AB - BA + B^2$.

15. If

$$A = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix},$$

calculate A^2 and verify that A satisfies $A^2 - 2A - 8I_2 = 0_2$.

16. Find a matrix

$$A = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

such that

$$A^2 + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = I_3.$$

17. If

$$A = \begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix},$$

determine all values of x and y for which $A^2 = A$.

18. The Pauli spin matrices σ_1, σ_2, and σ_3 are defined by

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

and

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Verify that they satisfy

$$\sigma_1\sigma_2 = i\sigma_3, \qquad \sigma_2\sigma_3 = i\sigma_1, \qquad \sigma_3\sigma_1 = i\sigma_2.$$

If A and B are $n \times n$ matrices, we define their **commutator**, denoted $[A, B]$, by

$$[A, B] = AB - BA.$$

Thus, $[A, B] = 0$ if and only if A and B commute. That is, $AB = BA$. Problems 19–22 require the commutator.

19. If

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix},$$

find $[A, B]$.

20. If

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

compute all of the commutators $[A_i, A_j]$, and determine which of the matrices commute.

21. If

$$A_1 = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \qquad A_2 = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$
$$A_3 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

verify that $[A_1, A_2] = A_3$, $[A_2, A_3] = A_1$, $[A_3, A_1] = A_2$.

22. If A, B and C are $n \times n$ matrices, find $[A, [B, C]]$ and prove the *Jacobi identity*

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

23. Use the index form of the matrix product to prove properties (2.2.4) and (2.2.5).

24. Prove parts 1 and 2 of Theorem 2.2.21.

25. Prove property 2 of the identity matrix.

26. If A and B are $n \times n$ matrices, prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

27. If

$$A = \begin{bmatrix} 1 & -1 & 1 & 4 \\ 2 & 0 & 2 & -3 \\ 3 & 4 & -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ -1 & 2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix},$$

find A^T, B^T, AA^T, AB and $B^T A^T$.

28. Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix}$, and let S be the matrix with column vectors

$$\mathbf{s}_1 = \begin{bmatrix} -x \\ 0 \\ x \end{bmatrix}, \qquad \mathbf{s}_2 = \begin{bmatrix} -y \\ y \\ -y \end{bmatrix}, \qquad \mathbf{s}_3 = \begin{bmatrix} z \\ 2z \\ z \end{bmatrix},$$

where x, y, z are constants.

(a) Show that $AS = [\mathbf{s}_1, \mathbf{s}_2, 7\mathbf{s}_3]$.

(b) Find all values of x, y, z such that $S^T AS = \operatorname{diag}(1, 1, 7)$.

29. A matrix that is a multiple of I_n is called an $n \times n$ **scalar matrix**.

(a) Determine the 4×4 scalar matrix whose trace is 8.

(b) Determine the 3×3 scalar matrix such that the product of the elements on the main diagonal is 343.

30. Prove that for each positive integer n, there is a unique scalar matrix whose trace is a given constant k.

If A is an $n \times n$ matrix, then the matrices S and T defined by

$$S = \tfrac{1}{2}(A + A^T), \qquad T = \tfrac{1}{2}(A - A^T)$$

are referred to as the symmetric and skew-symmetric parts of A, respectively. Problems 31–34 investigate properties of S and T.

31. Use the properties of the transpose to show that S and T are symmetric and skew-symmetric, respectively.

32. Find S and T for the matrix

$$A = \begin{bmatrix} 1 & -5 & 3 \\ 3 & 2 & 4 \\ 7 & -2 & 6 \end{bmatrix}.$$

33. If A is an $n \times n$ symmetric matrix, show that $T = 0$. What is the corresponding result for skew-symmetric matrices?

34. Show that every $n \times n$ matrix can be written as the sum of a symmetric and a skew-symmetric matrix.

35. Prove that if A is an $n \times p$ matrix and $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, then DA is the matrix obtained by multiplying the ith row vector of A by $d_i (1 \le i \le n)$.

36. Use the properties of the transpose to prove that

(a) AA^T is a symmetric matrix.

(b) $(ABC)^T = C^T B^T A^T$.

For Problems 37–40, determine the derivative of the given matrix function.

37. $A(t) = \begin{bmatrix} e^{-2t} \\ \sin t \end{bmatrix}$.

38. $A(t) = \begin{bmatrix} t & \sin t \\ \cos t & 4t \end{bmatrix}$.

39. $A(t) = \begin{bmatrix} e^t & e^{2t} & t^2 \\ 2e^t & 4e^{2t} & 5t^2 \end{bmatrix}$.

40. $A(t) = \begin{bmatrix} \sin t & \cos t & 0 \\ -\cos t & \sin t & t \\ 0 & 3t & 1 \end{bmatrix}$.

41. Let $A = [a_{ij}(t)]$ be an $m \times n$ matrix function and let $B = [b_{ij}(t)]$ be an $n \times p$ matrix function. Use the definition of matrix multiplication to prove that

$$\frac{d}{dt}(AB) = A\frac{dB}{dt} + \frac{dA}{dt}B.$$

For Problems 42–45, determine $\int_a^b A(t)\, dt$ for the given matrix function.

42. $A(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$, $a = 0, b = \pi/2$.

43. $A(t) = \begin{bmatrix} e^t & e^{-t} \\ 2e^t & 5e^{-t} \end{bmatrix}$, $a = 0, b = 1$.

44. $A(t) = \begin{bmatrix} e^{2t} & \sin 2t \\ t^2 - 5 & te^t \\ \sec^2 t & 3t - \sin t \end{bmatrix}$, $a = 0$, $b = 1$.

45. The matrix function $A(t)$ in Problem 39, with $a = 0$ and $b = 1$.

Integration of matrix functions given in the text was done with definite integrals, but one can naturally compute indefinite integrals of matrix functions as well, by performing indefinite integrals for each element of the matrix function.

In Problems 46–49, evaluate the indefinite integral $\int A(t)\, dt$ for the given matrix function. You may assume that the constants of all indefinite integrations are zero.

46. $A(t) = \begin{bmatrix} 2t \\ 3t^2 \end{bmatrix}$.

47. The matrix function $A(t)$ in Problem 40.

48. The matrix function $A(t)$ in Problem 43.

49. The matrix function $A(t)$ in Problem 44.

2.3 Terminology for Systems of Linear Equations

As we mentioned in Section 2.1, a main aim of this chapter is to apply matrices to determine the solution properties of any system of linear equations. We are now in a position to pursue that aim. We begin by introducing some notation and terminology.

DEFINITION 2.3.1

The general $m \times n$ system of linear equations is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned} \tag{2.3.1}$$

where the **system coefficients** a_{ij} and the **system constants** b_j are given scalars and $x_1, x_2, \ldots, x_n$ denote the unknowns in the system. If $b_i = 0$ for all i, then the system is called **homogeneous**; otherwise it is called **nonhomogeneous**.

DEFINITION 2.3.2

By a **solution** to the system (2.3.1) we mean an ordered n-tuple of scalars, $(c_1, c_2, \ldots, c_n)$, which, when substituted for $x_1, x_2, \ldots, x_n$ into the left-hand side of system (2.3.1), yield the values on the right-hand side. The set of all solutions to system (2.3.1) is called the **solution set** to the system.

Remarks

1. Usually the a_{ij} and b_j will be real numbers, and we will then be interested in determining only the real solutions to system (2.3.1). However, many of the problems that arise in the later chapters will require the solution to systems with complex coefficients, in which case the corresponding solutions will also be complex.
2. If $(c_1, c_2, \ldots, c_n)$ is a solution to the system (2.3.1), we will sometimes specify this solution by writing $x_1 = c_1, x_2 = c_2, \ldots, x_n = c_n$. For example, the ordered pair of numbers $(1, 2)$ is a solution to the system

$$\begin{aligned} x_1 + x_2 &= 3, \\ 3x_1 - 2x_2 &= -1, \end{aligned}$$

and we could express this solution in the equivalent form $x_1 = 1, x_2 = 2$.