For Problems 9–12, determine the solution set to $A\mathbf{x} = \mathbf{b}$, and show that all solutions are of the form (4.9.3).

**9.** $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & 9 \\ 1 & 5 & 21 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 11 \\ 10 \end{bmatrix}$.

**10.** $A = \begin{bmatrix} 2 & -1 & 1 & 4 \\ 1 & -1 & 2 & 3 \\ 1 & -2 & 5 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 6 \\ 13 \end{bmatrix}$.

**11.** $A = \begin{bmatrix} 1 & 1 & -2 \\ 3 & -1 & -7 \\ 1 & 1 & 1 \\ 2 & 2 & -4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -3 \\ 2 \\ 0 \\ -6 \end{bmatrix}$.

**12.** $A = \begin{bmatrix} 1 & 1 & -1 & 5 \\ 0 & 2 & -1 & 7 \\ 4 & 2 & -3 & 13 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

**13.** Show that a $3 \times 7$ matrix $A$ with $\text{nullity}(A) = 4$ must have $\text{colspace}(A) = \mathbb{R}^3$. Is $\text{rowspace}(A) = \mathbb{R}^3$?

**14.** Show that a $6 \times 4$ matrix $A$ with $\text{nullity}(A) = 0$ must have $\text{rowspace}(A) = \mathbb{R}^4$. Is $\text{colspace}(A) = \mathbb{R}^4$?

**15.** Prove that if $\text{rowspace}(A) = \text{nullspace}(A)$, then $A$ contains an even number of columns.

**16.** Show that a $5 \times 7$ matrix $A$ must have $2 \leq \text{nullity}(A) \leq 7$. Give an example of a $5 \times 7$ matrix $A$ with $\text{nullity}(A) = 2$ and an example of a $5 \times 7$ matrix $A$ with $\text{nullity}(A) = 7$.

**17.** Show that $3 \times 8$ matrix $A$ must have $5 \leq \text{nullity}(A) \leq 8$. Give an example of a $3 \times 8$ matrix $A$ with $\text{nullity}(A) = 5$ and an example of a $3 \times 8$ matrix $A$ with $\text{nullity}(A) = 8$.

**18.** Prove that if $A$ and $B$ are $n \times n$ matrices and $A$ is invertible, then

$$\text{nullity}(AB) = \text{nullity}(B).$$

[**Hint:** $B\mathbf{x} = \mathbf{0}$ if and only if $AB\mathbf{x} = \mathbf{0}$.]

## 4.10 The Invertible Matrix Theorem II

In Section 2.8, we gave a list of characterizations of invertible matrices (Theorem 2.8.1). In view of the concepts introduced in this chapter, we are now in a position to add to the list that was begun there.

**Theorem 4.10.1** **(Invertible Matrix Theorem)**

Let $A$ be an $n \times n$ matrix with real elements. The following conditions on $A$ are equivalent:

**(a)** $A$ is invertible.

**(h)** $\text{nullity}(A) = 0$.

**(i)** $\text{nullspace}(A) = \{\mathbf{0}\}$.

**(j)** The columns of $A$ form a linearly independent set of vectors in $\mathbb{R}^n$.

**(k)** $\text{colspace}(A) = \mathbb{R}^n$ (that is, the columns of $A$ span $\mathbb{R}^n$).

**(l)** The columns of $A$ form a basis for $\mathbb{R}^n$.

**(m)** The rows of $A$ form a linearly independent set of vectors in $\mathbb{R}^n$.

**(n)** $\text{rowspace}(A) = \mathbb{R}^n$ (that is, the rows of $A$ span $\mathbb{R}^n$).

**(o)** The rows of $A$ form a basis for $\mathbb{R}^n$.

**(p)** $A^T$ is invertible.

**Proof** The equivalence of (a) and (h) follows at once from Theorem 2.8.1(d) and the Rank-Nullity Theorem (Theorem 4.9.1). The equivalence of (h) and (i) is immediately clear. The equivalence of (a) and (j) is immediate from Theorem 2.8.1(c) and Theorem 4.5.14. Since the dimension of $\text{colspace}(A)$ is simply $\text{rank}(A)$, the equivalence of (a) and (k) is immediate from Theorem 2.8.1(d). Next, from the definition of a basis, we see that (j) and (k) are logically equivalent to (l). Moreover, since the row space and column space of $A$ have the same dimension, (k) and (n) are equivalent. Since $\text{rowspace}(A) = \text{colspace}(A^T)$, the equivalence of (k) and (n) proves that (a) and (p) are equivalent. Finally, the equivalence of (a) and (p) proves that (j) is equivalent to (m) and that (l) is equivalent to (o). ■

**Example 4.10.2** Do the rows of the matrix below span $\mathbb{R}^4$?

$$A = \begin{bmatrix} -2 & -2 & 1 & 3 \\ 3 & 3 & 0 & -1 \\ -1 & -1 & -2 & -5 \\ 2 & 2 & 1 & 1 \end{bmatrix}$$

**Solution:** We see by inspection that the columns of $A$ are linearly dependent, since the first two columns are identical. Therefore, by the equivalence of (j) and (n) in the Invertible Matrix Theorem, the rows of $A$ do not span $\mathbb{R}^4$. □

**Example 4.10.3** If $A$ is an $n \times n$ matrix such that the linear system $A^T\mathbf{x} = \mathbf{0}$ has no nontrivial solution $\mathbf{x}$, then $\text{nullspace}(A^T) = \{\mathbf{0}\}$, and thus $A^T$ is invertible by the equivalence of (a) and (i) in the Invertible Matrix Theorem. Thus, by the same theorem, we can conclude that the columns of $A$ form a linearly independent set. □

Despite the lengthy list of characterizations of invertible matrices that we have been able to develop so far, this list is still by no means complete. In the next chapter, we will use linear transformations and eigenvalues to provide further characterizations of invertible matrices.

## Exercises for 4.10

### Skills

- Be well familiar with all of the conditions (a)–(p) in the Invertible Matrix Theorem that characterize invertible matrices.

### True-False Review

For Questions 1–10, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

**1.** The set of all row vectors of an invertible matrix is linearly independent.

**2.** An $n \times n$ matrix can have $n$ linearly independent rows and $n$ linearly dependent columns.

**3.** The set of all row vectors of an $n \times n$ matrix can be linearly dependent while the set of all columns is linearly independent.

**4.** If $A$ is an $n \times n$ matrix with $\det(A) = 0$, then the columns of $A$ must form a basis for $\mathbb{R}^n$.

**5.** If $A$ and $B$ are row-equivalent $n \times n$ matrices such that $\text{rowspace}(A) \neq \mathbb{R}^n$, then $\text{colspace}(B) \neq \mathbb{R}^n$.

**6.** If $E$ is an $n \times n$ elementary matrix and $A$ is an $n \times n$ matrix with $\text{nullspace}(A) = \{\mathbf{0}\}$, then $\det(EA) = 0$.

**7.** If $A$ and $B$ are $n \times n$ invertible matrices, then $\text{nullity}([A|B]) = 0$, where $[A|B]$ is the $n \times 2n$ matrix with the blocks $A$ and $B$ as shown.

**8.** A matrix of the form

$$\begin{bmatrix} 0 & a & 0 \\ b & 0 & c \\ 0 & d & 0 \end{bmatrix}$$

cannot be invertible.

**9.** A matrix of the form

$$\begin{bmatrix} 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & e & 0 & f \\ g & 0 & h & 0 \end{bmatrix}$$

cannot be invertible.

**10.** A matrix of the form

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

such that $ae - bd = 0$ cannot be invertible.

## 4.11 Inner Product Spaces

We now extend the familiar idea of a dot product for geometric vectors to an arbitrary vector space $V$. This enables us to associate a magnitude with each vector in $V$ and also to define the angle between two vectors in $V$. The major reason that we want to do this is that, as we will see in the next section, it enables us to construct orthogonal bases in a vector space, and the use of such a basis often simplifies the representation of vectors. We begin with a brief review of the dot product.

Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ be two arbitrary vectors in $\mathbb{R}^3$, and consider the corresponding geometric vectors

$$\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}, \qquad \mathbf{y} = y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}.$$

The dot product of $\mathbf{x}$ and $\mathbf{y}$ can be defined in terms of the components of these vectors as

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3. \tag{4.11.1}$$

An equivalent geometric definition of the dot product is

$$\mathbf{x} \cdot \mathbf{y} = ||\mathbf{x}||\ ||\mathbf{y}|| \cos\theta, \tag{4.11.2}$$

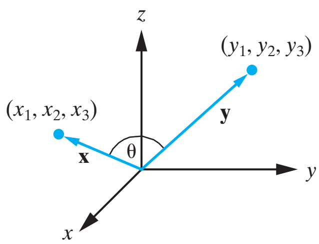


**Figure 4.11.1:** Defining the dot product in $\mathbb{R}^3$.

where $||\mathbf{x}||$, $||\mathbf{y}||$ denote the lengths of $\mathbf{x}$ and $\mathbf{y}$ respectively, and $0 \leq \theta \leq \pi$ is the angle between them. (See Figure 4.11.1.)

Taking $\mathbf{y} = \mathbf{x}$ in Equations (4.11.1) and (4.11.2) yields

$$||\mathbf{x}||^2 = \mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 + x_3^2,$$

so that the length of a geometric vector is given in terms of the dot product by

$$||\mathbf{x}|| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Furthermore, from Equation (4.11.2), the angle between any two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ is

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{||\mathbf{x}||\ ||\mathbf{y}||}, \tag{4.11.3}$$

which implies that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal (perpendicular) if and only if

$$\mathbf{x} \cdot \mathbf{y} = 0.$$

In a general vector space, we do not have a geometrical picture to guide us in defining the dot product, hence our definitions must be purely algebraic. We begin by considering the vector space $\mathbb{R}^n$, since there is a natural way to extend Equation (4.11.1) in this case. Before proceeding, we note that from now on we will use the standard terms *inner product* and *norm* in place of dot product and length, respectively.