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1. Exam 1

1. (10 pts) Determine a lower bound for the radius of convergence of series solutions

$$(x^2-4x+5)y''+(x+3)y'+4(x^2-4x+5)y=0$$

about $x_0=1$.

2. Consider the differential equation

$$(x-2)^2(x+1)y''+3(x^2+x-6)y'+(4x+1)y=0.$$

(a) (8 pts) Show that $x_0=2$ is a regular singular point.
(b) (7 pts) Find the indicial equation of a series solution of the form

$$y=\phi(r,x)=\sum_{n=0}^{\infty}a_n(r)(x-2)^{r+n}$$

and also find the exponents at the singular point $x_0=2$.

3. Consider a series solution $y=\sum_{n=0}^{\infty}a_nx^n$ about $x_0=0$ of

$$y''-xy'-2y=0.$$

(a) (10 pts) Find the recurrence relation for a_n.
(b) (5 pts) Find a general formula for a_n.
(c) (5 pts) Find two linearly independent series solutions.

4. (15 pts) Use the Laplace transform to solve the initial value problem

$$y''-7y'+12y=0,\quad y(0)=1,\quad y'(0)=0.$$

5.

(a) (8 pts) Use the definition of the Laplace transform

$$\mathcal{L}\{f(t)\}=\int_0^{\infty}e^{-st}f(t)dt$$

to show that

$$\mathcal{L}\{e^{at}\}=\frac{1}{s-a}\qquad\text{for } s>a.$$

(b) (7 pts) Find the Laplace transform of

$$f(t)=\left[2(t-5)^2+\cos(t-5)+4\right]u_5(t).$$

6. (10 pts) Find the inverse Laplace transform of

$$F(s)=\frac{(s+1)e^{-7s}}{s^2-6s+13}.$$

7. Consider the initial value problem

$$(\dagger)\quad \phi'(t)-\int_0^t(t-\xi)^2\phi(\xi)d\xi=\delta(t-3),\quad \phi(0)=1.$$

(a) (8 pts) Convert the differential equation (†) to an algebraic equation in $\Phi(s)=\mathcal{L}\{\phi(t)\}$ (but do **not** solve the equation).
(b) (7 pts) Let $\phi(t)$ be the solution of the equation (†). Evaluate the following integral

$$\int_0^{\infty}e^{-st}(2\phi(t)+\sin 3t)dt.$$

2. Exam 1-Solution

1. (10 pts) Determine a lower bound for the radius of convergence of series solutions

$$(x^2-4x+5)y''+(x+3)y'+4(x^2-4x+5)y=0$$

about $x_0=1$.

Solution) Note that $x_0=1$ is an ordinary point and

$$p(x)=\frac{x+3}{x^2-4x+5},\quad q(x)=4.$$

The roots of $x^2-4x+5=0$ are $2+i$ and $2-i$. The distance from $x_0=1$ to the nearest root $2+i$ is $\sqrt{2}$ (you may take $2-i$ as well), and so the radius of convergence of $p(x)$ is $\rho_p=\sqrt{2}$. The radius of convergence of $q(x)$ is $\rho_q=\infty$. Therefore, $\min\{\sqrt{2},\infty\}=\sqrt{2}$ and so we find that the radius of convergence ρ of the series solution is at least $\sqrt{2}$, which is a lower bound.

2. Consider the differential equation

$$(x-2)^2(x+1)y''+3(x^2+x-6)y'+(4x+1)y=0.$$

(a) (8 pts) Show that $x_0=2$ is a regular singular point.

Solution) Clearly, $x_0=2$ is a singular point because

$$(x-2)^2(x+1)=0\implies x=2,-1.$$

Use $x^2+x-6=(x-2)(x+3)$ and compute

$$p_0=\lim_{x\to2}(x-2)p(x)=\lim_{x\to2}(x-2)\frac{3(x-2)(x+3)}{(x-2)^2(x+1)}=\lim_{x\to2}\frac{3(x+3)}{(x+1)}=5$$

and

$$q_0=\lim_{x\to2}(x-2)^2q(x)=\lim_{x\to2}(x-2)^2\frac{4x+1}{(x-2)^2(x+1)}=\lim_{x\to2}\frac{4x+1}{x+1}=3.$$

Therefore $x_0=1$ is a regular singular point.

(b) (7 pts) Find the indicial equation of a series solution of the form

$$y=\phi(r,x)=\sum_{n=0}^{\infty}a_n(r)(x-2)^{r+n}$$

and also find the exponents at the singular point $x_0=2$.

Solution) The indicial equation is

$$0=r(r-1)+p_0r+q_0=r(r-1)+5r+3\implies r^2+4r+3=0\implies r=-1,-3$$

and the exponents of singularity at $x_0=2$ are $r=-1,-3$.

3. Consider a series solution $y=\sum_{n=0}^{\infty}a_nx^n$ about $x_0=0$ of

$$y''-xy'-2y=0.$$

(a) (10 pts) Find the recurrence relation for a_n.
(b) (5 pts) Find a general formula for a_n.
(c) (5 pts) Find two linearly independent series solutions.

Solution) (a) Compute

$$y'=\sum_{n=1}^{\infty}na_nx^{n-1},\quad y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$$

and put y, y', y'' into the differential equation to find that

$$\begin{aligned} 0 &= y'' - xy' - 2y \\ &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - 2\sum_{n=0}^{\infty} a_n x^n \end{aligned}$$

Use the shifting formula $n \to n+2$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

to get

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n - 2\sum_{n=0}^{\infty} a_n x^n \\ &= (2a_2 - 2a_0) + \sum_{n=1}^{\infty}\Big[(n+2)(n+1)a_{n+2} - (n+2)a_n\Big] \end{aligned}$$

and

$$\begin{cases} a_2 = a_0 \\ (n+2)(n+1)a_{n+2} - (n+2)a_n = 0, \quad n \geq 1 \end{cases}$$

(b) The recurrence relation can be simplified to

$$a_{n+2} = \frac{1}{n+1}a_n, \quad n \geq 1.$$

Considering *even and odd* cases we see that

$$a_{2m} = \frac{a_0}{1 \cdot 3 \cdot 5 \cdots (2m-3) \cdot (2m-1)}, \quad m \geq 1$$

and

$$a_{2m+1} = \frac{a_1}{2 \cdot 4 \cdot 6 \cdots (2m-2) \cdot (2m)}, \quad m \geq 0.$$

(c) The general solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + \sum_{m=1}^{\infty} a_{2m}x^{2m} + \sum_{m=0}^{\infty} a_{2m+1}x^{2m+1} \\ &= a_0 + \sum_{m=1}^{\infty} \frac{a_0}{1 \cdot 3 \cdot 5 \cdots (2m-3) \cdot (2m-1)}x^{2m} + \sum_{m=0}^{\infty} \frac{a_1}{2 \cdot 4 \cdot 6 \cdots (2m-2) \cdot (2m)}x^{2m+1} \end{aligned}$$

and so

$$y_1(x) = 1 + \sum_{m=1}^{\infty} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2m-3) \cdot (2m-1)}x^{2m}$$

and

$$y_2(x) = \sum_{m=0}^{\infty} \frac{1}{2 \cdot 4 \cdot 6 \cdots (2m-2) \cdot (2m)}x^{2m+1}$$

are two linearly independent solutions.

4. (15 pts) Use the Laplace transform to solve the initial value problem

$$y'' - 7y' + 12y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution) Set $Y(s) = \mathcal{L}\{y(t)\}$ and compute the Laplace transform

$$\begin{aligned} 0 &= \mathcal{L}\{y'' - 7y' + 12y\} \\ &= \left[s^2 Y(s) - sy(0) - y'(0)\right] - 7\left[sY(s) - y(0)\right] + 12Y(s) \\ &= \left[s^2 Y(s) - s\right] - 7\left[sY(s) - 1\right] + 12Y(s) \\ &= (s^2 - 7s + 12)Y(s) - s + 7. \end{aligned}$$

Solve for $Y(s)$ and compute the partial fractions

$$Y(s) = \frac{s+3}{s^2 - 7s + 12} = \frac{s-7}{(s-3)(s-4)} = \frac{4}{s-3} - \frac{3}{s-4}.$$

Taking the inverse Laplace transform we find that

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} = 4e^{3t} - 3e^{4t}.$$

5.

(a) (8 pts) Use the definition of the Laplace transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

to show that

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad \text{for } s > a.$$

Solution) By the definition

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \lim_{A\to\infty} \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^A = \lim_{A\to\infty} \frac{e^{-(s-a)A}}{-(s-a)} + \frac{1}{s-a} = \frac{1}{s-a}$$

because $\lim_{A\to\infty} e^{-kA} = 0$ for $k = s - a > 0$.

(b) (7 pts) Find the Laplace transform of

$$f(t) = \left[2(t-5)^2 + \cos(t-5) + 4\right] u_5(t).$$

Solution) Note that

$$f(t) = h(t-5)u_5(t)$$

where

$$h(t) = 2t^2 + \cos t + 4 \implies H(s) = \mathcal{L}\{h(t)\} = \frac{4}{s^3} + \frac{s}{s^2+1} + \frac{4}{s}.$$

By the general formula

$$\mathcal{L}\{f(t)\} = e^{-5s} H(s) = e^{-5s}\left(\frac{4}{s^3} + \frac{s}{s^2+1} + \frac{4}{s}\right).$$

6. (10 pts) Find the inverse Laplace transform of

$$F(s) = \frac{(s+1)e^{-7s}}{s^2 - 6s + 13}.$$

Solution) Write

$$F(s) = \frac{(s+1)e^{-7s}}{s^2 - 6s + 13} = e^{-7s} H(s)$$

where

$$H(s) = \frac{(s+1)}{s^2 - 6s + 13} = \frac{s-3}{(s-3)^2 + 4} + 2\frac{2}{(s-3)^2 + 4} \implies h(t) = \mathcal{L}^{-1}\{H(s)\} = e^{3t}\cos 2t + 2e^{3t}\sin 2t$$

from $s^2 - 6s + 13 = (s-3)^2 + 4$. We see that

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{e^{-7s}H(s)\} = h(t-7)u_7(t) = \left[e^{3(t-7)}\cos 2(t-7) + 2e^{3(t-7)}\sin 2(t-7)\right] u_7(t).$$

7. Consider the initial value problem

$$(\dagger) \quad \phi'(t) - \int_0^t (t-\xi)^2 \phi(\xi) d\xi = \delta(t-3), \quad \phi(0) = 1.$$

(a) (8 pts) Convert the differential equation (†) to an algebraic equation in $\Phi(s) = \mathcal{L}\{\phi(t)\}$ (but do **not** solve the equation).

Solution) Let $\Phi(s) = \mathcal{L}\{\phi(t)\}$, $f(t) = t^2$ and $F(s) = \mathcal{L}\{f(t)\} = \frac{2}{s^3}$. Note that we may rewrite the equation (†) as follows:

$$\phi'(t) - (f * \phi)(t) = \delta(t-3).$$

Take the Laplace transforms on both sides to see

$$\mathcal{L}\{\phi'(t)\} + \mathcal{L}\{(f*\phi)(t)\} = \mathcal{L}\{\delta(t-3)\} \implies (s\Phi(s) - \phi(0)) - F(s)\Phi(s) = e^{-3s} \implies \left(s - \frac{2}{s^3}\right)\Phi(s) = 1 + e^{-3s}.$$

(b) (7 pts) Let $\phi(t)$ be the solution of the equation (†). Evaluate the following integral

$$\int_0^\infty e^{-st}(2\phi(t) + \sin 3t)dt.$$

Solution) Note that

$$\int_0^\infty e^{-st}(2\phi(t) + \sin 3t)dt = 2\mathcal{L}\{\phi(t)\} + \mathcal{L}\{\sin 3t\} = 2\Phi(s) + \frac{3}{s^2+9} = \frac{2s^3(1+e^{-3s})}{s^4-1} + \frac{3}{s^2+9}$$

because

$$\left(s - \frac{2}{s^3}\right)\Phi(s) = 1 + e^{-3s} \implies \Phi(s) = \frac{s^3(1+e^{-3s})}{s^4-2}$$

from (a).