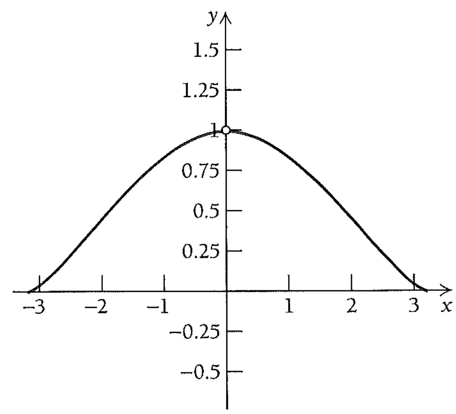
If the function $f(x) = \frac{\sin x}{x}$ were continuous at 0, then it would be simple to compute the limit at 0; we would just substitute $x = 0$. However, since the denominator is x, $f(x)$ is not even defined at $x = 0$. We investigate this limit numerically and graphically.

Limit Numerically

x	$\frac{\sin x}{x}$
−0.5	0.95885
−0.4	0.97355
−0.3	0.98507
−0.2	0.99335
−0.1	0.99833
−0.01	0.99998
0.01	0.99998
0.1	0.99833
0.2	0.99335
0.3	0.98507
0.4	0.97355
0.5	0.95885

Limit Graphically



> **Technology Connection**
>
> **Cosine Limit**
>
> Use the **TABLE** function and the graph to determine
>
> $$\lim_{x \to 0} \frac{\cos x - 1}{x}.$$
>
> Explain why we cannot use the continuity properties to determine this limit.

We can see from both the table and the graph that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

The other trigonometric limit needed in Section 2.5 is

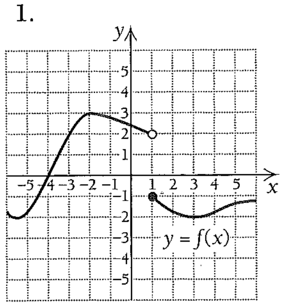
$$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0.$$

This limit is investigated in the Technology Connection located in the margin, and it is investigated algebraically in Section 2.2.

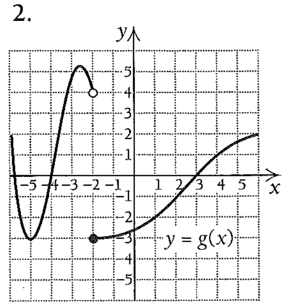
Exercise Set 2.1

Determine whether each of the following is continuous.

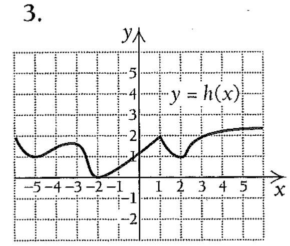
1.



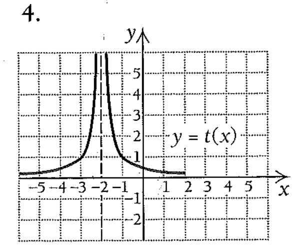
2.



3.



4.



Use the graphs and functions in Exercises 1–4 to answer each of the following.

5. a) Find $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 1^-} f(x)$, and $\lim_{x\to 1} f(x)$.
   b) Find $f(1)$.
   c) Is f continuous at $x = 1$?
   d) Find $\lim_{x\to -2} f(x)$.
   e) Find $f(-2)$.
   f) Is f continuous at $x = -2$?

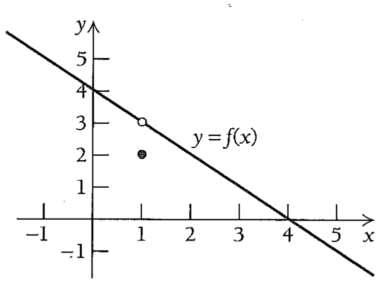
6. a) Find $\lim_{x\to 1^+} g(x)$, $\lim_{x\to 1^-} g(x)$, and $\lim_{x\to 1} g(x)$.
   b) Find $g(1)$.
   c) Is g continuous at $x = 1$?
   d) Find $\lim_{x\to -2} g(x)$.
   e) Find $g(-2)$.
   f) Is g continuous at $x = -2$?

7. a) Find $\lim_{x\to 1} h(x)$.
   b) Find $h(1)$.
   c) Is h continuous at $x = 1$?
   d) Find $\lim_{x\to -2} h(x)$.
   e) Find $h(-2)$.
   f) Is h continuous at $x = -2$?

8. a) Find $\lim_{x\to 1} t(x)$.
   b) Find $t(1)$.
   c) Is t continuous at $x = 1$?
   d) Find $\lim_{x\to -2} t(x)$.
   e) Find $t(-2)$.
   f) Is t continuous at $x = -2$?

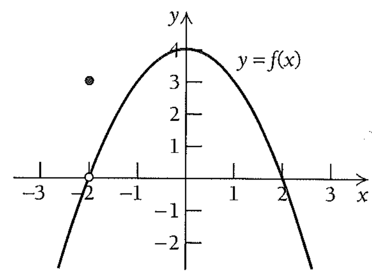
In Exercises 9–12, use the graphs to find the limits and answer the related questions.

9. Consider $f(x) = \begin{cases} 4 - x, & \text{for } x \neq 1, \\ 2, & \text{for } x = 1. \end{cases}$



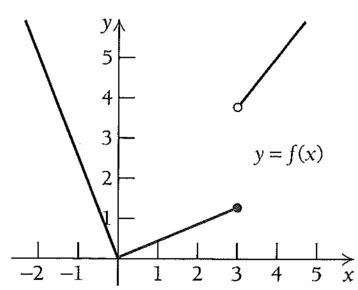
   a) Find $\lim_{x\to 1^+} f(x)$.
   b) Find $\lim_{x\to 1^-} f(x)$.
   c) Find $\lim_{x\to 1} f(x)$.
   d) Find $f(1)$.
   e) Is f continuous at $x = 1$?
   f) Is f continuous at $x = 2$?

10. Consider $f(x) = \begin{cases} 4 - x^2, & \text{for } x \neq -2, \\ 3, & \text{for } x = -2. \end{cases}$



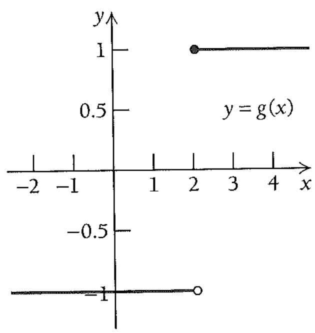
   a) Find $\lim_{x\to -2^+} f(x)$.
   b) Find $\lim_{x\to -2^-} f(x)$.
   c) Find $\lim_{x\to -2} f(x)$.
   d) Find $f(-2)$.
   e) Is f continuous at $x = -2$?
   f) Is f continuous at $x = 1$?

11. Refer to the graph of f below to determine whether each statement is true or false.



   a) $\lim_{x\to 0^+} f(x) = f(0)$
   b) $\lim_{x\to 0^-} f(x) = f(0)$
   c) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$
   d) $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x)$
   e) $\lim_{x\to 0} f(x)$ exists.
   f) $\lim_{x\to 3} f(x)$ exists.
   g) f is continuous at $x = 0$.
   h) f is continuous at $x = 3$.

12. Refer to the graph of g below to determine whether each statement is true or false.



a) $\lim_{x \to 2^+} g(x) = g(2)$

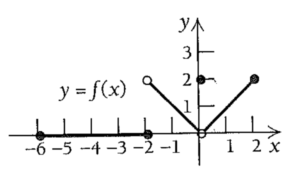
b) $\lim_{x \to 2^-} g(x) = g(2)$

c) $\lim_{x \to 2^+} g(x) = \lim_{x \to 2^-} g(x)$

d) $\lim_{x \to 2} g(x)$ exists.

e) g is continuous at $x = 2$.

13. Refer to the graph of f below to determine whether each statement is true or false.



a) $\lim_{x \to -2^+} f(x) = 1$

b) $\lim_{x \to -2^-} f(x) = 0$

c) $\lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x)$

d) $\lim_{x \to -2} f(x)$ exists.

e) $\lim_{x \to -2} f(x) = 2$

f) $\lim_{x \to 0} f(x) = 0$

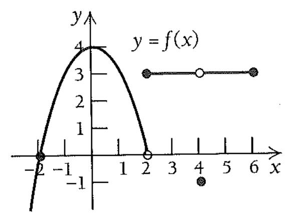
g) $f(0) = 2$

h) f is continuous at $x = -2$.

i) f is continuous at $x = 0$.

j) f is continuous at $x = -1$.

14. Refer to the graph of f below to determine whether each statement is true or false.



a) $\lim_{x \to 2^-} f(x) = 3$

b) $\lim_{x \to 2^+} f(x) = 0$

c) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

d) $\lim_{x \to 2} f(x)$ exists.

e) $\lim_{x \to 4} f(x)$ exists.

f) $\lim_{x \to 4} f(x) = f(4)$

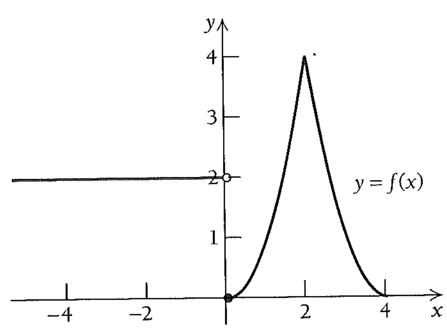
g) f is continuous at $x = 4$.

h) f is continuous at $x = 0$.

i) $\lim_{x \to 3} f(x) = \lim_{x \to 5} f(x)$

j) f is continuous at $x = 2$.

15. Refer to the graph of f below to determine whether each statement is true or false.



a) $\lim_{x \to 0^+} f(x) = f(0)$

b) $\lim_{x \to 0^-} f(x) = f(0)$

c) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$

d) $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$

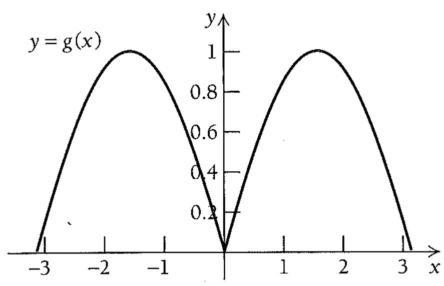
e) $\lim_{x \to 0} f(x)$ exists.

f) $\lim_{x \to 2} f(x)$ exists.

g) f is continuous at $x = 0$.

h) f is continuous at $x = 2$.

16. Refer to the graph of g below to determine whether each statement is true or false.



a) $\lim_{x \to 0^+} g(x) = g(0)$

b) $\lim_{x \to 0^-} g(x) = g(0)$

c) $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^-} g(x)$

d) $\lim_{x \to 0} g(x)$ exists.

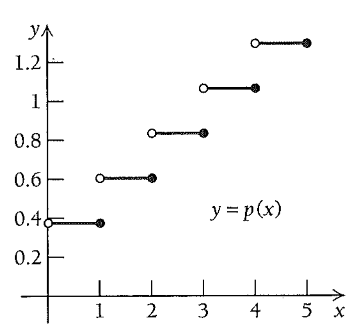
e) g is continuous at $x = 0$.

APPLICATIONS

The Postage Function. Postal rates are \$0.37 for the first ounce and \$0.23 for each additional ounce (or fraction thereof). Formally speaking, if x is the weight of a letter in ounces, then $p(x)$ is the cost of mailing the letter, where

$$\begin{aligned} p(x) &= \$0.37, && \text{if } 0 < x \le 1, \\ p(x) &= \$0.60, && \text{if } 1 < x \le 2, \\ p(x) &= \$0.83, && \text{if } 2 < x \le 3, \end{aligned}$$

and so on, up to 13 oz (at which point postal cost also depends on distance). The graph of p is shown below.



17. Is p continuous at 1? at 1.5? at 2? at 2.01?

18. Is p continuous at 2.99? at 3? at 3.04? at 4?

Using the graph of the postage function, find each of the following limits, if it exists.

19. $\lim_{x \to 1^-} p(x)$, $\lim_{x \to 1^+} p(x)$, $\lim_{x \to 1} p(x)$

20. $\lim_{x \to 2^-} p(x)$, $\lim_{x \to 2^+} p(x)$, $\lim_{x \to 2} p(x)$

21. $\lim_{x \to 2.6^-} p(x)$, $\lim_{x \to 2.6^+} p(x)$, $\lim_{x \to 2.6} p(x)$

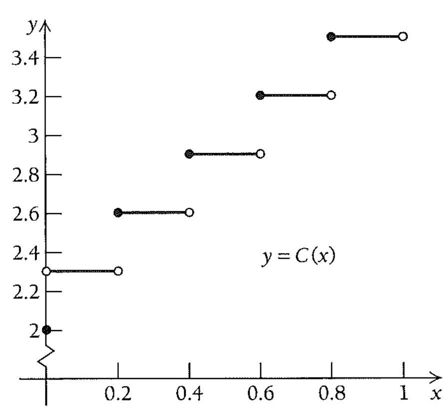
22. $\lim_{x \to 3} p(x)$

23. $\lim_{x \to 3.4} p(x)$

Taxicab Fares. In New York City, taxicabs charge passengers \$2.00 for entering a cab and then \$0.30 for each one-fifth of a mile (or fraction thereof) that the cab travels. (There are additional charges for slow traffic and idle times, but these are not considered in this problem.) Formally speaking, if x is the distance traveled in miles, then $C(x)$ is the cost of the taxi fare, where

$$\begin{aligned} C(x) &= \$2.00, && \text{if } x = 0, \\ C(x) &= \$2.30, && \text{if } 0 < x < 0.2, \\ C(x) &= \$2.60, && \text{if } 0.2 \le x < 0.4, \\ C(x) &= \$2.90, && \text{if } 0.4 \le x < 0.6, \end{aligned}$$

and so on. The graph of C is shown below.



24. Is C continuous at 0.1? at 0.2? at 0.25? at 0.267?

25. Is C continuous at 2.3? at 2.5? at 2.6? at 3.0?

Using the graph of the taxicab fare function, find each of the following limits, if it exists.

26. $\lim_{x \to 1/4^-} C(x)$, $\lim_{x \to 1/4^+} C(x)$, $\lim_{x \to 1/4} C(x)$

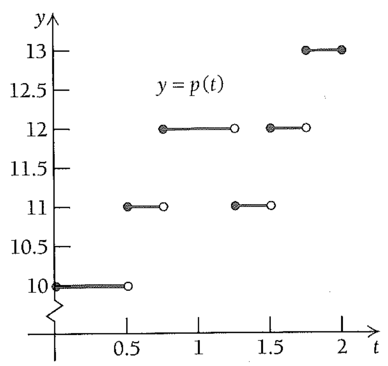
27. $\lim_{x \to 0.2^-} C(x)$, $\lim_{x \to 0.2^+} C(x)$, $\lim_{x \to 0.2} C(x)$

28. $\lim_{x \to 0.6^-} C(x)$, $\lim_{x \to 0.6^+} C(x)$, $\lim_{x \to 0.6} C(x)$

29. $\lim_{x \to 0.5} C(x)$

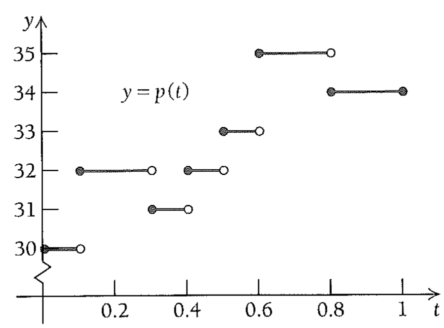
30. $\lim_{x \to 0.4} C(x)$

Population Growth. In a certain habitat, the deer population as a function of time (measured in years) is given in the graph of p below.



31. Identify each point where the population function is discontinuous.
32. At each point where the function is not continuous, identify an event that might have occurred in the population to cause the discontinuity.
33. Find $\lim_{t \to 1.5^+} p(t)$.
34. Find $\lim_{t \to 1.5^-} p(t)$.

Population Growth. The population of bears in a certain region is given by the graph of p below. Time t is measured in months.

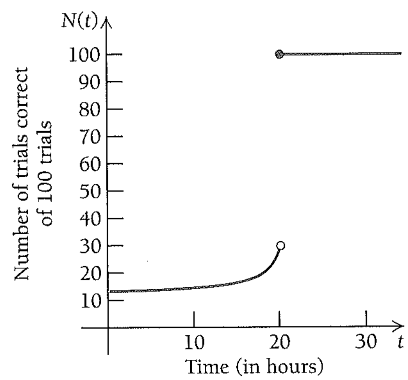


35. Identify each point where the population function is discontinuous.
36. At each point where the function is not continuous, identify an event that might have occurred in the population to cause the discontinuity.
37. Find $\lim_{t \to 0.6^+} p(t)$.
38. Find $\lim_{t \to 0.6^-} p(t)$.

A Learning Curve. In psychology one often takes a certain amount of time t to learn a task. Suppose that the goal is to do a task perfectly and that you are practicing the ability to master it. After a certain time period, what is known to psychologists as an "I've got it!" experience occurs, and you are able to perform the task perfectly.

tw 39. At what point do you think the "I've got it!" experience happens on the learning curve below?

tw 40. Why do you think the curve below is constant for inputs $t \geq 20$?



Using the graph above, find each of the following limits, if it exists.

41. $\lim_{t \to 20^+} N(t)$, $\lim_{t \to 20^-} N(t)$, $\lim_{t \to 20} N(t)$
42. $\lim_{t \to 30^-} N(t)$, $\lim_{t \to 30^+} N(t)$, $\lim_{t \to 30} N(t)$
43. Is N continuous at 20? at 30?
44. Is N continuous at 10? at 26?

SYNTHESIS

tw 45. Discuss three ways in which a function may not be continuous at a point a. Draw graphs to illustrate your discussion.

Use the Continuity Properties C1–C5 to justify that the function is continuous. Then give the limit using the fact that the function is continuous.

46. $f(x) = x^2 + 5x - 5$; $\lim_{x \to 3} f(x)$
47. $f(x) = 3x^3 + 2x^2 - 9x + 4$; $\lim_{x \to 1} f(x)$
48. $g(x) = \dfrac{x}{x-1}$, for $x \neq 1$; $\lim_{x \to -1} \dfrac{x}{x-1}$
49. $g(x) = \dfrac{x^2 + 9x - 7}{x + 2}$,

    for $x \neq -2$; $\lim_{x \to 1} \dfrac{x^2 + 9x - 7}{x + 2}$
50. $\tan x$, for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$; $\lim_{x \to \pi/4} \tan x$
51. $\cot x$, for $0 < x < \pi$; $\lim_{x \to \pi/3} \cot x$
52. $\sec x$, for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$; $\lim_{x \to \pi/6} \sec x$
53. $\csc x$, for $0 < x < \pi$; $\lim_{x \to \pi/4} \csc (x)$

54. $f(x) = \sqrt{x^2 + 2x + 4}$ (*Hint:* f is a composition of functions); $\lim_{x \to 3} f(x)$.

55. $f(x) = \sqrt{\sin x}$, for $0 < x < \pi$; $\lim_{x \to \pi/3} \sqrt{\sin x}$

56. $g(x) = \sin^2 x$; $\lim_{x \to \pi/4} \sin^2 x$

57. $g(x) = \cos(2x + 3\pi)$; $\lim_{x \to \pi/6} \cos(2x + 3\pi)$

Technology Connection

In Exercises 58–63, use a grapher to determine the limits. Make a table for each and draw the graph.

58. $\lim_{x \to 5} \dfrac{x^2 + 3x - 40}{x^2 + 4x - 45}$

59. $\lim_{x \to -1} \dfrac{x^3 + x^2 - x - 1}{x^3 + 1}$

60. $\lim_{x \to 0} \dfrac{1 - \cos x}{x}$

61. $\lim_{h \to 0} \dfrac{\tan h}{h}$

62. $\lim_{x \to 0} \dfrac{\sin x - x}{x^3}$

63. $\lim_{h \to 0} \dfrac{\csc h - 1}{h}$

2.2 Limits: Algebraically

OBJECTIVE

- Find limits using algebraic methods.

Using Limit Principles

If a function is continuous at a, we can substitute to find the limit.

EXAMPLE 1 Find $\lim_{x \to 0} \sqrt{x^2 - 3x + 2}$.

Solution Using the Continuity Principles, we have shown that polynomials such as $x^2 - 3x + 2$ are continuous for all values of x. When we restrict x to values for which $x^2 - 3x + 2$ is nonnegative, it follows from Principle C5 that $\sqrt{x^2 - 3x + 2}$ is continuous. Since $x^2 - 3x + 2$ is nonnegative when $x = 0$, we can substitute to find the limit:

$$\lim_{x \to 0} \sqrt{x^2 - 3x + 2} = \sqrt{0^2 - 3 \cdot 0 + 2} = \sqrt{2}.$$

EXAMPLE 2 Find $\lim_{x \to 2} \sin(x^2 - 4)$.

Solution By the Continuity Principles, we know that $x^2 - 4$ is continuous. Therefore, the composition $\sin(x^2 - 4)$ is also continuous since the sine function is continuous. To find the limit, we simply substitute $x = 2$.

$$\lim_{x \to 2} \sin(x^2 - 4) = \sin(2^2 - 4) = \sin(0) = 0.$$

Using the fact that many of the usual functions from algebra and trigonometry are continuous, we can compute many limits by simply evaluating the function at the point in question. It is also possible to use Limit Principles to compute limits. These principles can be used when we are uncertain of the continuity of the function.

Exercise Set 2.2

Find the limit using the algebraic method. Verify using the numerical or graphical method.

1. $\lim_{x \to 1} (x^2 - 3)$
2. $\lim_{x \to 1} (x^2 + 4)$
3. $\lim_{x \to 0} \frac{3}{x}$
4. $\lim_{x \to 0} \frac{-4}{x}$
5. $\lim_{x \to 3} (2x + 5)$
6. $\lim_{x \to 4} (5 - 3x)$
7. $\lim_{x \to -5} \frac{x^2 - 25}{x + 5}$
8. $\lim_{x \to -4} \frac{x^2 - 16}{x + 4}$
9. $\lim_{x \to -2} \frac{5}{x}$
10. $\lim_{x \to -5} \frac{-2}{x}$
11. $\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2}$
12. $\lim_{x \to -4} \frac{x^2 - x - 20}{x + 4}$

Find the limit. Use the algebraic method.

13. $\lim_{x \to 5} \sqrt[3]{x^2 - 17}$
14. $\lim_{x \to 2} \sqrt{x^2 + 5}$
15. $\lim_{x \to \pi/4} (x + \sin x)$
16. $\lim_{x \to \pi/6} (\cos x + \tan x)$
17. $\lim_{x \to 0} \frac{1 + \sin x}{1 - \sin x}$
18. $\lim_{x \to 0} \frac{1 + \cos x}{\cos x}$
19. $\lim_{x \to 2} \frac{1}{x - 2}$
20. $\lim_{x \to 1} \frac{1}{(x - 1)^2}$
21. $\lim_{x \to 2} \frac{3x^2 - 4x + 2}{7x^2 - 5x + 3}$
22. $\lim_{x \to -1} \frac{4x^2 + 5x - 7}{3x^2 - 2x + 1}$
23. $\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4}$
24. $\lim_{x \to 4} \frac{x^2 - 16}{x^2 - x - 12}$
25. $\lim_{h \to 0} (6x^2 + 6xh + 2h^2)$
26. $\lim_{h \to 0} (10x + 5h)$
27. $\lim_{h \to 0} \frac{-2x - h}{x^2(x + h)^2}$
28. $\lim_{h \to 0} \frac{-5}{x(x + h)}$
29. $\lim_{x \to 0} \frac{\tan x}{x}$
30. $\lim_{x \to 0} x \csc x$

SYNTHESIS

Find the limit, if it exists.

31. $\lim_{h \to 0} \frac{\sin x \sin h}{h}$
32. $\lim_{h \to 0} \frac{\sin x \cos h - \sin x}{h}$
33. $\lim_{x \to 0} \frac{x^2 + 3x}{x^2 - 2x^4}$
34. $\lim_{x \to 0} \frac{x^2 - 2x^4}{x^2 + 3x}$
35. $\lim_{x \to 0^+} \frac{x\sqrt{x}}{x + x^2}$
36. $\lim_{x \to 0} \frac{x + x^2}{x\sqrt{x}}$
37. $\lim_{x \to 2} \frac{x - 2}{x^2 - x - 2}$
38. $\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$
39. $\lim_{x \to 3} \frac{x^2 - 9}{2x - 6}$
40. $\lim_{x \to -2} \frac{3x^2 + 5x - 2}{x^2 - 3x - 10}$

Technology Connection

Further Use of the TABLE Feature. In Section 2.1, we discussed how to use the TABLE feature to find limits. Consider

$$\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x}.$$

Input–output tables for this function are shown below. The table on the left uses TblStart $= -1$ and ΔTbl $=$ 0.5. By using smaller and smaller step values and beginning closer to 0, we can refine the table and obtain a better estimate of the limit. On the right is an input–output table with TblStart $= -0.03$ and ΔTbl $= 0.01$.

x	y
−1	1
−0.5	0.585786
0	ERROR
0.5	0.449490
1	0.414214
1.5	0.387426
2	0.366025

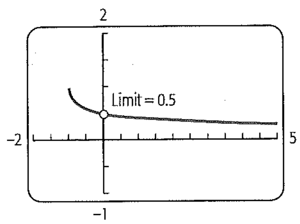
x	y
−0.03	0.503807
−0.02	0.502525
−0.01	0.501256
0	ERROR
0.01	0.498756
0.02	0.497525
0.03	0.496305

It appears that the limit is 0.5. We can verify this by graphing

$$y = \frac{\sqrt{1+x} - 1}{x}$$

and tracing the curve near $x = 0$, zooming in on that portion of the curve.

$y = \dfrac{\sqrt{1+x} - 1}{x}$



We see that

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = 0.5.$$

This can be verified algebraically. (*Hint*: Multiply by 1, using $\dfrac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1}$.)

In Exercises 41–48, find the limit. Use the **TABLE** feature and start with $\Delta\text{Tbl} = 0.1$. Then move to 0.01, 0.001, and 0.0001. When you think you know the limit, graph and use the **TRACE** feature to verify your assertion. Then try to verify algebraically.

41. $\displaystyle\lim_{a \to -2} \frac{a^2 - 4}{\sqrt{a^2 + 5} - 3}$

42. $\displaystyle\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1}$

43. $\displaystyle\lim_{x \to 0} \frac{\sqrt{3 - x} - \sqrt{3}}{x}$

44. $\displaystyle\lim_{x \to 0} \frac{\sqrt{4 + x} - \sqrt{4 - x}}{x}$

45. $\displaystyle\lim_{x \to 1} \frac{x - \sqrt[4]{x}}{x - 1}$

46. $\displaystyle\lim_{x \to 0} \frac{\sqrt{7 + 2x} - \sqrt{7}}{x}$

47. $\displaystyle\lim_{x \to 4} \frac{2 - \sqrt{x}}{4 - x}$

48. $\displaystyle\lim_{x \to 0} \frac{7 - \sqrt{49 - x^2}}{x}$

2.3 Average Rates of Change

OBJECTIVES

- Compute an average rate of change.
- Find a simplified difference quotient.

Let's say that a car travels 110 mi in 2 hr. Its *average rate of change* (*speed*) is 110 mi/2 hr, or 55 mi/hr (55 mph). On the other hand, suppose that you are on the freeway and you begin accelerating. Glancing at the speedometer, you see that at that *instant* your *instantaneous rate of change* is 55 mph. These are two quite different concepts. The first you are probably familiar with. The second involves ideas of limits and calculus. To understand *instantaneous rate of change,* we first use this section to develop a solid understanding of *average rate of change.*

Photosynthesis is the conversion of light energy to chemical energy that is stored in glucose or other organic compounds.[1] In the process, oxygen is produced. The following graph approximates the amount of oxygen a plant produces by photosynthesis. Time 0 is taken to be 8 A.M. The rate of photosynthesis depends on a number of factors, including the amount of light and temperature.

[1]N. A. Campbell and J. B. Reece, *Biology,* 6th ed. (Benjamin Cummings, New York 2002).

Exercise Set 2.3

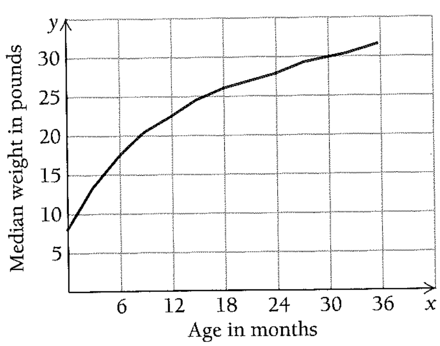
For the functions in each of Exercises 1–12, (a) find a simplified form of the difference quotient and (b) complete the following table.

x	h	$\dfrac{f(x+h)-f(x)}{h}$
4	2	
4	1	
4	0.1	
4	0.01	

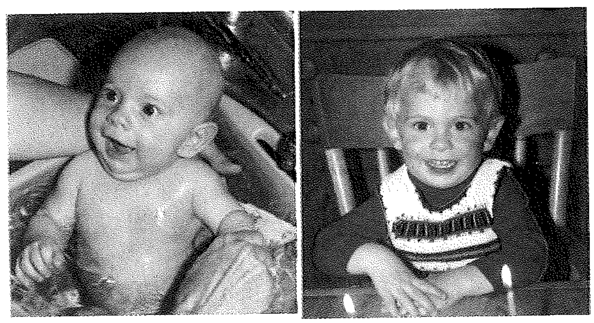
1. $f(x) = 7x^2$
2. $f(x) = 5x^2$
3. $f(x) = -7x^2$
4. $f(x) = -5x^2$
5. $f(x) = 7x^3$
6. $f(x) = 5x^3$
7. $f(x) = \dfrac{5}{x}$
8. $f(x) = \dfrac{4}{x}$
9. $f(x) = -2x + 5$
10. $f(x) = 2x + 3$
11. $f(x) = x^2 - x$
12. $f(x) = x^2 + x$

APPLICATIONS

13. *Growth of a Baby.* The median weight of boys is given in the graph below. Use the graph to estimate:[2]



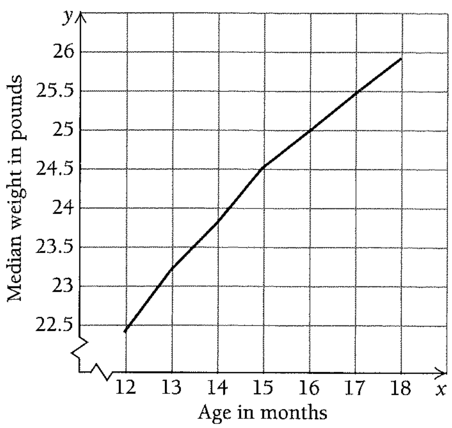
a) The average growth rate of a typical boy during his first year of life. (Your answer should be in pounds per month.)
b) The average growth rate of a typical boy during his second year of life.
c) The average growth rate of a typical boy during his third year of life.
d) The average growth rate of a typical boy during his first 3 yr of life.
e) When does the graph indicate that a boy's growth rate is greatest during his first 3 yr of life?



14. *Growth of a Baby.* Use the graph in Exercise 13 to estimate:
a) The average growth rate of a typical boy during his first 9 mo of life. (Your answer should be in pounds per month.)
b) The average growth rate of a typical boy during his first 6 mo of life.
c) The average growth rate of a typical boy during his first 3 mo of life.
tw d) Based on your answers in parts (a)–(c) and the graph, estimate what the average growth rate of a typical boy should be the first few weeks of his life.

[2]Centers for Disease Control. Developed by the National Center for Health Statistics in collaboration with the National Center for Chronic Disease Prevention and Health Promotion (2000).

15. *Growth of a Baby.* Use the following graph to estimate:



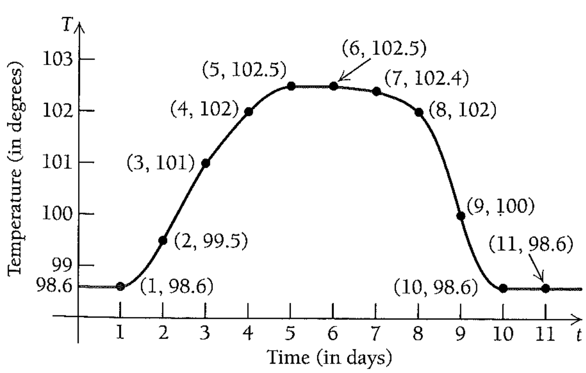
a) The average growth rate of a typical boy between ages 12 mo and 18 mo. (Your answer should be in pounds per month.)

b) The average growth rate of a typical boy between ages 12 mo and 14 mo.

c) The average growth rate of a typical boy between ages 12 mo and 13 mo.

tw d) Based on your answers in parts (a)–(c) and the graph, estimate the average growth rate of a typical boy when he is 12 mo old.

16. *Temperature During an Illness.* The temperature T, in degrees Fahrenheit, of a patient during an illness is shown in the following graph, where t is the time, in days.



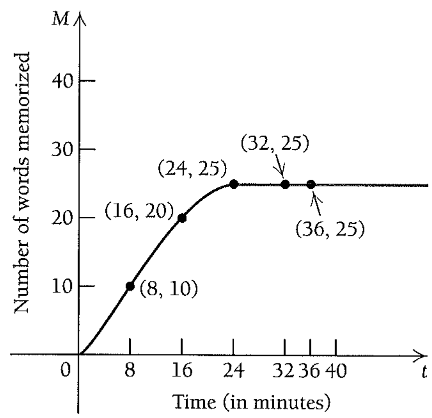
a) Find the average rate of change of T as t changes from 1 to 10. Using this rate of change, would you know that the person was sick?

b) Find the rate of change of T with respect to t, as t changes from 1 to 2; from 2 to 3; from 3 to 4; from 4 to 5; from 5 to 6; from 6 to 7; from 7 to 8; from 8 to 9; from 9 to 10; from 10 to 11.

c) When do you think the temperature began to rise? reached its peak? began to subside? was back to normal?

tw d) Explain your answers to part (c).

17. *Memory.* The total number of words $M(t)$ that a person can memorize in time t, in minutes, is shown in the following graph.



a) Find the average rate of change of M as t changes from 0 to 8; from 8 to 16; from 16 to 24; from 24 to 32; from 32 to 36.

tw b) Why do the average rates of change become 0 after 24 min?

18. *Average Velocity.* A car is at a distance s, in miles, from its starting point in t hours, given by

$$s(t) = 10t^2.$$

a) Find $s(2)$ and $s(5)$.

b) Find $s(5) - s(2)$. What does this represent?

c) Find the average rate of change of distance with respect to time as t changes from $t_1 = 2$ to $t_2 = 5$. This is known as **average velocity,** or **speed.**

19. *Average Velocity.* An object is dropped from a certain height. It is known that it will fall a distance s, in feet, in t seconds, given by

$$s(t) = 16t^2.$$

a) How far will the object fall in 3 sec?

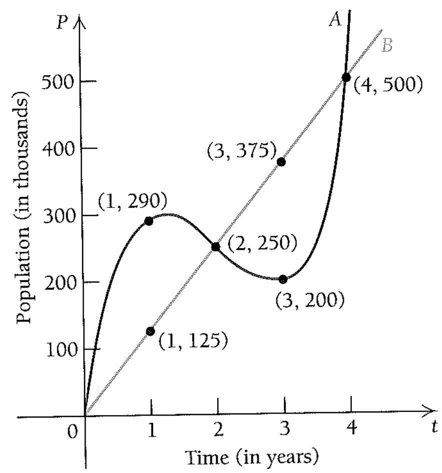
b) How far will the object fall in 5 sec?

c) What is the average rate of change of distance with respect to time during the period from 3 to 5 sec? This is also the *average velocity.*

20. *Gas Mileage.* At the beginning of a trip, the odometer on a car reads 30,680 and the car has a full tank of gas. At the end of the trip, the odometer reads 30,970. It takes 15 gal of gas to refill the tank.

a) What is the average rate of change of the number of miles with respect to the number of gallons?
b) What is the average rate of consumption (that is, the rate of change of the number of miles with respect to the number of gallons)?

21. *Population Growth.* The two curves shown in the following figure describe the number of people in each of two countries at time t, in years.



a) Find the average rate of change of each population (the number of people in the population) with respect to time t as t changes from 0 to 4. This is often called an **average growth rate.**

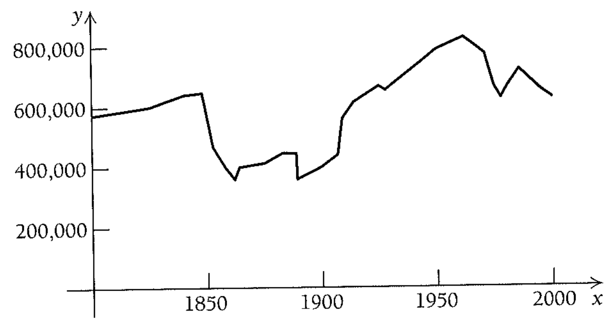
tw b) If the calculation in part (a) were the only one made, would we detect the fact that the populations were growing differently? Explain.

c) Find the average rates of change of each population as t changes from 0 to 1; from 1 to 2; from 2 to 3; from 3 to 4.

tw d) For which population does the statement "the population grew by 125 million each year" convey the least information about what really took place? Explain.

SYNTHESIS

Deer Population. The deer population in California from 1800 to 2000 is approximated in the graph shown below. Use this graph to answer Exercises 22 and 23.[3]



tw 22. Consider the parts of the graph from 1850 to 1860 and from 1890 to 1960. Discuss the differences between these two pieces of the graph in as many ways as you can. Be sure to consider average rates of change.

tw 23. Pick out pieces of the graph where the slopes and shapes are similar and pieces where the slopes and shapes are different. Explain the differences and the similarities. Can you identify historical occurrences that correspond to when the graph changes?

Find the simplified difference quotient.

24. $f(x) = mx + b$

25. $f(x) = ax^2 + bx + c$

26. $f(x) = ax^3 + bx^2$

27. $f(x) = \sqrt{x}$

$\left(\textit{Hint}: \text{Multiply by 1 using } \dfrac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}.\right)$

28. $f(x) = x^4$

29. $f(x) = \dfrac{1}{x^2}$

30. $f(x) = \dfrac{1}{1 - x}$

31. $f(x) = \dfrac{x}{1 + x}$

32. $f(x) = \sqrt{3 - 2x}$

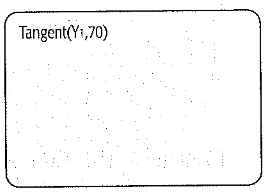
33. $f(x) = \dfrac{1}{\sqrt{x}}$

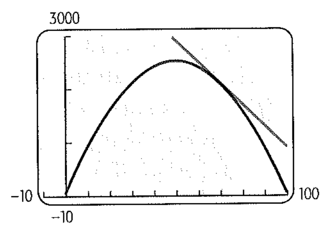
34. $f(x) = \dfrac{2x}{x - 1}$

[3]California Department of Fish and Game.

Technology Connection *(continued)*

To draw a tangent line at $x = 70$, we go to the home screen and select the TANGENT feature from the DRAW menu. Then we enter Tangent(Y1, 70). We see the graph of $f(x)$ and the tangent line at $x = 70$.





EXERCISES

For each of the following functions, evaluate the derivative at the given point. Then draw the graph and the tangent line.

1. $f(x) = x(100 - x)$;
$x = 20, x = 37, x = 50, x = 90$

2. $f(x) = -\frac{1}{3}x^3 + 6x^2 - 11x - 50$;
$x = -5, x = 0, x = 7, x = 12, x = 15$

3. $f(x) = 6x^2 - x^3$;
$x = -2, x = 0, x = 2, x = 4, x = 6.3$

4. $f(x) = x\sqrt{4 - x^2}$;
$x = -2, x = -1.3, x = -0.5, x = 0, x = 1, x = 2$

tw **5.** For the function in Exercise 4, try to draw a tangent line at $x = 3$ and estimate the derivative. What goes wrong? Explain.

CAUTION Some calculators will give answers for derivatives even though they do not exist. For example, $f(x) = |x|$ does not have a derivative at $x = 0$, but some calculators will give 0 for the answer. Try entering nDeriv(abs(X), X, 0) to see what your calculator does.

Exercise Set 2.4

In Exercises 1–16:

a) Graph the function.
b) Draw tangent lines to the graph at points whose x-coordinates are -2, 0, and 1.
c) Find $f'(x)$ by determining $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.
d) Find $f'(-2)$, $f'(0)$, and $f'(1)$. How do these slopes compare with those of the lines you drew in part (b)?

1. $f(x) = 5x^2$
2. $f(x) = 7x^2$
3. $f(x) = -5x^2$
4. $f(x) = -7x^2$
5. $f(x) = x^3$
6. $f(x) = -x^3$
7. $f(x) = 2x + 3$
8. $f(x) = -2x + 5$
9. $f(x) = -4x$
10. $f(x) = \frac{1}{2}x$
11. $f(x) = x^2 + x$
12. $f(x) = x^2 - x$
13. $f(x) = 2x^2 + 3x - 2$
14. $f(x) = 5x^2 - 2x + 7$
15. $f(x) = \dfrac{1}{x}$
16. $f(x) = \dfrac{5}{x}$
17. Find $f'(x)$ for $f(x) = mx$.
18. Find $f'(x)$ for $f(x) = ax^2 + bx + c$.

19. Find an equation of the tangent line to the graph of $f(x) = x^2$ at the point $(3, 9)$, at $(-1, 1)$, and at $(10, 100)$. See Example 3.

20. Find an equation of the tangent line to the graph of $f(x) = x^3$ at the point $(-2, -8)$, at $(0, 0)$, and at $(4, 64)$. See Example 4.

21. Find an equation of the tangent line to the graph of $f(x) = 5/x$ at the point $(1, 5)$, at $(-1, -5)$, and at $(100, 0.05)$. See Exercise 16.

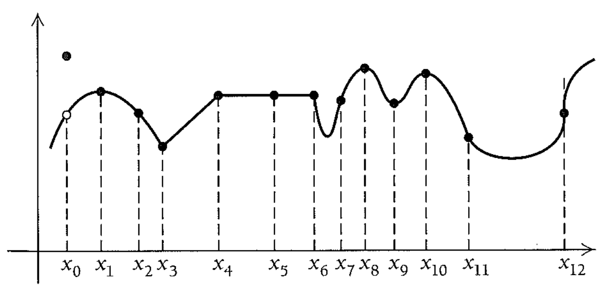
22. Find an equation of the tangent line to the graph of $f(x) = 2/x$ at the point $(-1, -2)$, at $(2, 1)$, and at $\left(10, \frac{1}{5}\right)$.

23. Find an equation of the tangent line to the graph of $f(x) = 4 - x^2$ at the point $(-1, 3)$, at $(0, 4)$, and at $(5, -21)$.

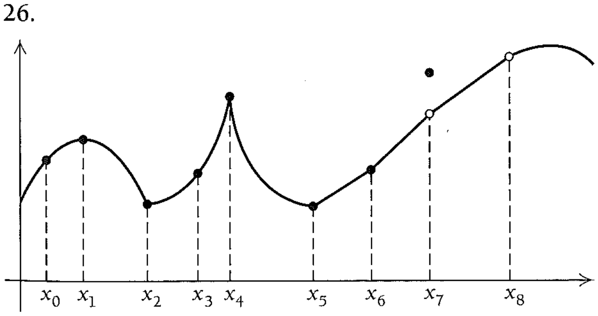
24. Find an equation of the tangent line to the graph of $f(x) = x^2 - 2x$ at the point $(-2, 8)$, at $(1, -1)$, and at $(4, 8)$.

List the points in the graph at which each function is not differentiable.

25.



26.



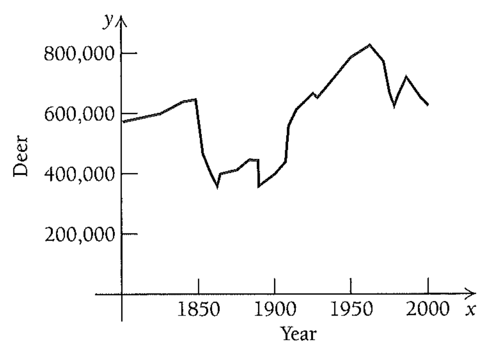
APPLICATIONS

27. *The Postage Function.* Consider the postage function defined on page 83. At what values is the function not differentiable?

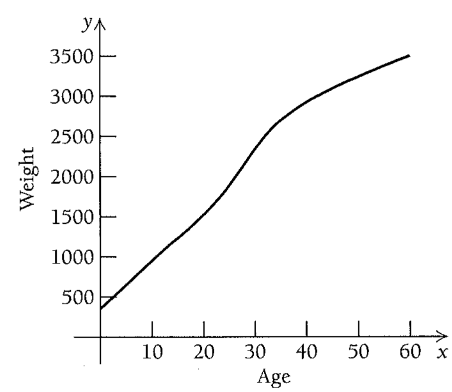
28. *The Taxicab Function.* Consider the taxicab function defined on page 83. At what values is the function not differentiable?

29. *Baseball Ticket Prices.* Consider the model for major league baseball average ticket prices in Exercise 35 of Exercise Set 1.2. At what values is the function not differentiable?

tw 30. *Deer Population.* Using the graph below to model the population of deer in California, in which years is the function not differentiable? Explain.



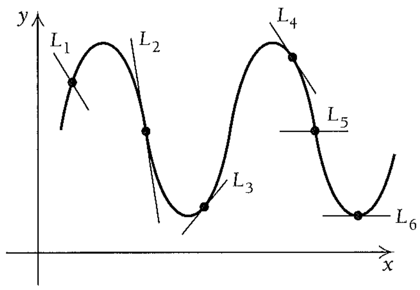
31. *Growth of Whales.* The graph below approximates the weight, in pounds, of a killer whale (*Orcinus orca*).[4] The age is given in months. Find all months where the weight function is not differentiable.



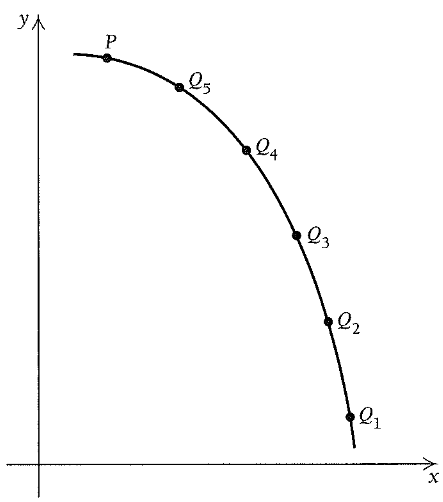
[4]SeaWorld.

SYNTHESIS

tw 32. Which of the following appear to be tangent lines? Try to explain why or why not.



tw 33. In the following figure, use a blue colored pencil and draw each secant line from point P to the points Q. Then use a red colored pencil and draw a tangent line to the curve at P. Describe what happens.



Find $f'(x)$.

34. $f(x) = x^4$

35. $f(x) = \dfrac{1}{x^2}$

36. $f(x) = \dfrac{1}{1 - x}$

37. $f(x) = \dfrac{x}{1 + x}$

38. $f(x) = \sqrt{x}$

$\left(\text{Multiply by 1, using } \dfrac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}.\right)$

39. $f(x) = \dfrac{1}{\sqrt{x}}$

40. $f(x) = \dfrac{3x}{x+5}$

41. Consider the function f given by

$$f(x) = \frac{x^2 - 9}{x + 3}.$$

For what values is this function not differentiable?

Technology Connection

42.–47. Use a grapher to do the numerical differentiation and draw the tangent lines in each of Exercises 19–24.

48. *Growth of a Baby.* The median weight w of a girl whose age t is between 0 and 36 mo can be approximated by the function

$$w(t) = 0.0006t^3 - 0.0484t^2 + 1.61t + 7.6,$$

where t is measured in months and w is measured in pounds. Use this approximation to make the following computations for a girl with median weight.[5]

Note: Some graphers use only the variables x and y, so you may need to change the variables when entering the function.

a) Graph w over the interval $[0, 36]$.
b) Find the equation of the secant line passing through the points $(12, w(12))$, and $(36, w(36))$. Then sketch the secant line using the same axes as in part (a).
c) Find the average rate of growth in pounds per month for a girl of median weight between ages 12 mo and 36 mo.
d) Repeat parts (b) and (c) for pairs of points $(12, w(12))$, and $(24, w(24))$; $(12, w(12))$ and $(18, w(18))$; $(12, w(12))$ and $(15, w(15))$.
e) What appears to be the slope of the tangent line at the point $(12, w(12))$?

[5]Centers for Disease Control. Developed by the National Center for Health Statistics in collaboration with the National Center for Chronic Disease Prevention and Health Promotion (2000).