This is a project consisting of several steps which you can follow and thus get an insight on the relation between the complex derivative and the advanced calculus notion of (real) differentiability

1. Let $f = u + iv$ be complex-valued near $z = z_0$. Then the derivative $f'(z_0)$ exists if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

2a. Let $u(x, y)$ be defined in a neighborhood of $z_0 = (x_0, y_0)$. Then u is *differentiable* at z_0 if (let $z_0 = x_0 + iy_0$)

$$(*) \quad u(x,y) - u(z_0) = A(x - x_0) + B(y - y_0) + \varepsilon(z)$$

where

$$\frac{\varepsilon(z)}{|z - z_0|} \to 0$$

b. Show that if u is differentiable at z_0 then the partials u_x, u_y exist at z_0

In $(*)$, set $y = y_0$, let $x \to x_0$. Then $\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} = A + \frac{\varepsilon(z)}{|x - x_0|} \to A$. So $u_x = A$; similarly, $u_y = B$.

c. (i) Find an example of a function $u(x, y)$ which is differentiable at $(0, 0)$ and for which the partial derivatives exist only at $(0, 0)$. (This has to be a contrived example.)

Here is one example:

$$u(x,y) = x^2 + y^2 \quad \text{(both } x \text{ and } y \text{ rational or both irrational)}$$
$$= -x^2 - y^2 \quad \text{(one of } x, y \text{ rational, the other irrational)}$$

Note that u is continuous only at $(0,0)$, and

$$u_x = u_y(0,0) = 0.$$

At $(0,0)$ we have

$$u(x,y) - u(0,0) = \varepsilon(x,y)$$

where $|\varepsilon(x,y)| = x^2 + y^2$,

and of course

$$\frac{\varepsilon(x,y)}{|z - z_0|} \stackrel{1}{=} \frac{x^2 + y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} \to 0 \text{ as } (x,y) \to (0,0)$$

Since u is continuous nowhere else, it can't be differentiable anywhere else either.

3a. Let $f'(z_0)$ exist. Show that this means that u and v are differentiable at z_0. (Hint: write out the difference quotient defining f' and consider real and imaginary parts.) $\frac{f(z)-f(z_0)}{z-z_0} = \frac{u(z)-u(z_0)+i(v(z)-v(z_0))}{z-z_0} \to A+iB$, say,

as $z \to z_0$. In other words:

(*) $\Delta f = \Delta u + i\Delta v = (A+iB+\varepsilon+i\eta)(\Delta x+i\Delta y)$ where A and B are constants and ε and η (both real) $\to 0$ as $\Delta z \to 0$.

If we take real part: (1) $\Delta u = A\Delta x - B\Delta y + \varepsilon'\Delta x - \eta\Delta y$

for imaginary parts, (2) $\Delta v = A\Delta y + B\Delta x + (\varepsilon'\Delta y + \eta\Delta x)$.

This means u and v are differentiable. For example, if we refer to 2a for Δu, we find

$\varepsilon(z) = \varepsilon'\Delta x - \eta\Delta y$, so that

$\frac{\varepsilon(z)}{|z-z_0|} \le \frac{\varepsilon'|\Delta x|}{|\Delta z|} + \frac{\eta|\Delta y|}{|\Delta y|} < \varepsilon' + \eta \to 0 \ (\Delta z \to 0)$

b. Show that u and v satisfy the Cauchy-Riemann equations at z_0. (This was done in class!)

This is easy from (*) in 3a. We know (see me if you can't verify this) that A and B are constants. And in (1) $A = u_x$, $-B = u_y$; in (2) $A = v_y$ (since it multiplies Δy) and $B = v_x$. So $u_x = v_y$; $u_y = -v_x$

c. Show that the *converse* is true: if u and v are differentiable at z_0 and satisfy the Cauchy-Riemann equations there, then $f'(z_0)$ exists (this was done in class but you should make a good copy for your notes)

Let's look at (*) but now we know that $A = u_x$ and $B = v_x$. So (*) becomes

$$\Delta f = \Delta u + i\Delta v = (u_x + iv_x)\Delta z + (\varepsilon' + i\eta)\Delta z.$$

Let's divide by Δz

$$\frac{\Delta f}{\Delta z} = u_x + iv_x + (\varepsilon' + i\eta).$$

where u_x and v_x are partial derivatives evaluated at z_0 and $\varepsilon', \eta \to 0$. Thus

$$f'(z) = u_x + iv_x \ (= v_y - iu_y \text{ by C-R}).$$