MA525 ON CAUCHY'S THEOREM AND GREEN'S THEOREM

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(EDITED BY JOSIAH YODER)

1. Introduction

No doubt the most important result in this course is Cauchy's theorem. Every critical theorem in the course takes advantage of it, and it is even used to show that all analytic functions must have derivatives of all orders.[1]

There are many ways to formulate it, but the most simple, direct and useful is this: *Let f be analytic inside and on the simple closed curve γ. Then*

$$\int_\gamma f(z)dz = 0. \tag{1.1}$$

The most natural way to prove this is by using Green's theorem. We state the conclusion of Green's theorem now, leaving a discussion of the hypotheses and proof for later. The formula reads: *D is a region bounded by a system of curves γ (oriented in the 'positive' direction with respect to D) and P and Q are functions defined on $D \cup \gamma$. Then*

$$\int_\gamma Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy. \tag{1.2}$$

Green's theorem leads to a trivial proof of Cauchy's theorem. Although this is only a formal[2] proof, since we have not discussed the conditions necessary to apply Green's theorem, I think it is impressive how 'simple' and natural the proof becomes:

$$f = u + iv \quad dz = dx + idy \tag{1.3}$$

and then[3]

$$\int_\gamma f(z)dz = \int_\gamma (u+iv)(dx+idy) = \int_\gamma u\,dx - v\,dy + i\int_\gamma u\,dy + v\,dx. \tag{1.4}$$

If we apply Green's theorem to each of these line integrals,

$$= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy, \tag{1.5}$$

and use the Cauchy-Riemann equations[4]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{1.6}$$

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[1] Recall that a complex function is *analytic* in a region if its first-order derivative exists.

[2] "formal" in the sense that we are applying the right steps without checking whether it is the right thing to do. This is somewhat like a formal circumstance such as meeting the president at a banquet and asking, "how do you do?" when you couldn't care less how he does. Asking "How do you do?" is the formal thing to do, but you probably haven't stopped to consider if it is really what you want to know. It's generally good to start formally, and then check the conditions.

[3] Recall that we define a complex integral along a contour as $\int_\gamma f(z)dz = \int_{t_0}^{t_1} f(z(t))(dz/dt)dt$ where $z(t)$ is a parameterization of the path γ. Thus $\int_\gamma (u+iv)(dx+idy) = \int_{t_0}^{t_1}(u+iv)((dx/dt)+i(dy/dt))\,dt$ is a standard definite integral and nothing to be afraid of! As an example, consider the integral on the left side of Green's Theorem: $\int_\gamma Pdx + Qdy = \int_{t_0}^{t_1} P(dx/dt) + Q(dy/dt)\,dt$. From this it is clear that we can split it into two integrals: $\int_\gamma Pdx + Qdy = \int_\gamma Pdx + \int_\gamma Qdy$.

[4] We will review a proof of the Cauchy-Riemann equations as part of Thm. 3 on page 5.

we see that the integrand in each double integral is (identically) zero. In this sense, Cauchy's theorem is an immediate consequence of Green's theorem.

In fact, Green's theorem is itself a fundamental result in mathematics — the fundamental theorem of calculus in higher dimensions. Proofs of Green's theorem are in all the calculus books, where it is always assumed that P and Q have *continuous partial derivatives*. Thus our simple proof would apply only to functions with continuous partial derivatives as well. Unless Cauchy's Theorem applies to all analytic functions, it cannot be used as the basis for the many important theorems derived from it in this course.

This note makes a case for the simple, elegant proof above by demonstrating that Green's theorem applies to *all* analytic functions, not just functions with continuous partial derivatives.

The heart of this proof is a variation on E. Goursat's 'elementary' proof of Cauchy's theorem. The other observations are not original either, but I am collecting them together for your convenience.

Editor's Note: Though this document is only 8 pages or so, it will probably take two sittings to read through it. I'll give you a warning when I think it's time for break!

2. What is wrong?

There are two possible objections to the proof I just presented. One, which we consider only briefly here, is that we have not carefully described what kind of curves we are allowing, or what we mean by the 'positive' direction of circuiting γ. We shall allow only rectangular regions where the positive direction may be clearly defined. Extending the proof to other regions is a problem of point-set topology or geometric measure theory, and this note offers no insight on these issues. *(We may eventually look at these questions in Appendix D).*

The second, and principle, objection is that we have not stated the hypotheses on P and Q needed to apply Green's theorem. Supplying these hypotheses, and a proof that Green's theorem still holds, is the purpose of this note.

As mentioned above, the proofs of Green's theorem in the calculus books assume that the partial derivatives are continuous. When applied to our analytic function $f(z)$, this means that we are assuming that the partial derivatives $u_x (= \partial u/\partial x)$, u_y, v_x and v_y are continuous. The partial derivatives of an analytic function are continuous, but this is something that is most often proved using Cauchy's Theorem. To avoid circular reasoning, a proof of Cauchy's Theorem should not make this assumption.

The purpose of this note is to show that we do not need to assume P and Q have continuous partials; indeed, Green's theorem holds when P and Q satisfy conditions which fit *exactly* with what it means for $f = P + iQ$ to be analytic.

3. Outline

Our goal is to prove a general form of Cauchy's Theorem:

Theorem 1. *Let f be analytic inside a rectangle R and continuous on its boundary. Then Cauchy's theorem (1.1) holds.*

We will do this using the techniques of Section 1 with a formulation of Green's Theorem which does not depend on continuous partial derivatives:

Theorem 2. *Let P and Q be differentiable inside and on a rectangle R with boundary γ and suppose that*

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \qquad (3.1)$$

Then Green's Theorem (1.2) holds.

The proof for this theorem will be presented in Section 8. Note that in principal $\partial Q/\partial x - \partial P/\partial y$ could be identically zero without the component terms $\partial Q/\partial x$ and $\partial P/\partial y$ being continuous.

To be able to use Theorem 2 to derive Theorem 1, we will check two things:

(1) That u and v are differentiable at each point z at which $f'(z)$ exists. To aid in this endeavor, we will review what it means to be differentiable.
(2) That $\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$. This follows from the Cauchy-Riemann equations, which we will also review.

Verifying these points will also prepare us for the final step, proving Green's Theorem (Thm 2). For this, we will use the same argument that E. Goursat introduced to give his famous 'elementary' proof of Cauchy's theorem, which appeared in Volume 1 of the *Transactions of the American Mathematical Society.*

4. How do we prove Thm 1?

Once we have proved Green's theorem, as stated in Thm. 2, we can apply the proof given in the first section of this paper. But how do we know the combination $\partial Q/\partial x - \partial P/\partial y$ is identically zero?

This is because the Cauchy-Riemann equations hold. To be specific, we know that if f is analytic, then $\partial u/\partial x - \partial v/\partial y = 0$. Similarly, $-\partial v/\partial x - \partial u/\partial y = 0$. Thus one of the conditions required for Thm. 2 is met perfectly by any analytic function. The remaining condition is that P and Q be differentiable.

If you are feeling a little uncertain about the Cauchy-Riemann equations, don't despair! We will actually prove these on the side in Section 6.

5. Differentiability

Our version of Green's Theorem requires that the P and Q be differentiable. Before we can show that the real and imaginary parts of any analytic function are differentiable, a little review of differentiability is in order. Although this is standard material in third-semester calculus, the review is likely helpful.

5.1. Differentiability in one dimension.

In one dimension, we define differentiability as having a derivative. Now the derivative exists as long as the limit that defines it exists:

$$\lim_{x\to a} \frac{u(x) - u(a)}{x - a} = A \tag{5.1}$$

But this limit is equivalent to the following limit:

$$\lim_{x\to a} \frac{u(x) - u(a) - A(x-a)}{x - a} = 0. \tag{5.2}$$

If either limit exists, the other is guaranteed to exist as well, either can be used as the definition of the derivative. The nice thing about the second limit is that it is zero. And there is a very convenient theorem involving limits which are zero. If we multiply this limit by anything with absolute value 1, it will still be zero. So let's multiply it by $(x-a)/|x-a|$. This yields

$$\lim_{x\to a} \frac{u(x) - u(a) - A(x-a)}{|x - a|} = 0. \tag{5.3}$$

Again, this limit exists if (and only if) the function $u(x)$ has a derivative at $x = a$. So we can use it as an alternate form for the derivative.

Why is this function nice? It has allowed us to do what we had hoped to do originally – to divide by something like $(x - a)$ in a higher-dimensional space.

For convenience, we will define $S(x) = u(x) - u(a) - A(x-a)$, so that if and only if $u(x)$ is differentiable, we have

$$\lim_{x\to a} \frac{S(x)}{|x-a|} = 0 \tag{5.4}$$

Intuitively, it is nice to put the definition of $S(x)$ into the form

$$u(x) = u(a) + A(x-a) + S(x) \tag{5.5}$$

where we can see that the original function is equal to its value at the point a, plus the straight line $A(x-a)$, plus the error term $S(x)$, which is a function which not only goes to 0 at $x = a$, but has its derivative there at well. $S(x)$ is truly an unobtrusive function in the vicinity of $x = a$!

5.2. **Differentiability in higher dimensions.** In dimensions greater than one, being differentiable is a much stronger property than having partial derivatives, since it is a condition on the entire region surrounding a point p, not just the paths of constant x or y through p. To see how this is, let us consider the definition of differentiability for a real-valued function u in the domain D.

Definition. The function $u(x,y)$ is *differentiable* at (a,b) if there are constants A and B so that

$$u(x,y) - u(a,b) = A\,(x-a) + B\,(y-b) + S(x,y) \tag{5.6}$$

where the 'remainder' S satisfies

$$\lim_{(x,y)\to(a,b)} \frac{S(x,y)}{|(x,y)-(a,b)|} = \lim_{(x,y)\to(a,b)} \frac{S(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0. \tag{5.7}$$

This definition is a simple extension of the one-dimensional derivative we developed above. Intuitively, this means that a function is differentiable if it can be locally approximated by a linear function with a remainder term that vanishes near the point (a,b). Again this vanishing term is well-behaved. Not only does it go to 0, it's derivative is 0 at (a,b) as well.

5.3. **Differentiable functions have partial derivatives.**

Definition. Based on this intuition, we might guess that every differentiable function has partial derivatives. Proving this makes a good exercise, as follows. If we set y identically equal to b, and let $x \to a$, then $\sqrt{(x-a)^2+(y-b)^2} = \sqrt{(x-a)^2} = |x-a|$, so (5.7) tells us that

$$\lim_{x\to a} \frac{u(x,b) - u(a,b) - A\cdot(x-a)}{|x-a|} = 0. \tag{5.8}$$

It is quite amazing how useful this formulation is. Since the limit on the right side is zero, we can multiply the left side by any factor of absolute value one without changing the equation. Multiplying by $|x-a|/(x-a)$, we have at once

$$\lim_{x\to a} \frac{u(x,b) - u(a,b) - A\cdot(x-a)}{(x-a)} = 0: \tag{5.9}$$

$$u_x(a,b) = A; \tag{5.10}$$

similarly, we see that $u_y(a,b) = B$. So it is not difficult to show that a differentiable function has partial derivatives.

However, as we mentioned at the beginning, being differentiable is a much stronger property than having partial derivatives, since it is a condition independent of how $(x,y) \to (a,b)$. For example, the function $u(x,y) = xy/(x^2+y^2)$ has $u_x(0,0) = u_y(0,0) = 0$, but it is not differentiable, or even continuous, at (0,0) since $u(x,x) = 1/2$ on the 45-degree line through the origin, but is zero on both the axes.

5.4. **Differentiable functions of complex variables.**

Definition. Before leaving this section, let's note that differentiability is defined similarly for functions of complex variables. The function $u(z) = u(x, y)$ is differentiable if there are constants A and B such that

$$u(z) - u(z_0) = A \cdot (x - a) + B \cdot (y - b) + S(z) \tag{5.11}$$

where the remainder S satisfies

$$\lim_{z \to z_0} \frac{S(z)}{|z - z_0|} = 0. \tag{5.12}$$

We can imagine this formula intuitively in the same way as the first formula, since $S(z)$ can be imagined as a surface. Because z is a complex number, its real and imaginary parts take the place of x and y in our previous discussion.

6. First Blood

Let us prove a little theorem. The proof is not hard at all, and if you go through it, you will see that you are just rearranging equalities everywhere. And for this proof, the moral of the story is critical to our argument; it says that for u and v to be differentiable is as natural as $f = u + iv$ having a derivative (that is, as natural as f being analytic), meeting the condition that P and Q are differentiable in Thm. 2.

Theorem 3. *Let $f = u + iv$ be defined in some neighborhood of $z_0 = a + ib$. Then f' exists at z_0 if and only if, at z_0, we have both that u and v are differentiable, and that the partials of u and v satisfy $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (the Cauchy-Riemann equations).*

Proof. First let's assume that the derivative exists and is $f'(z_0) = A + iB$. We show that u and v are differentiable and satisfy Cauchy-Riemann. Let's write the difference between the function and its local linear approximation $f(z) - f(z_0) - (A + iB) \cdot (z - z_0)$ in terms of u, v, x, and y. We also separate real and imaginary parts to yield:

$$\begin{aligned}
&f(z) - f(z_0) - (A + iB) \cdot (z - z_0) \\
&= (u(z) + iv(z)) - (u(z_0) + iv(z_0)) - (A + iB) \cdot ((x - a) + i(y - b)) \\
&= (u(z) - u(z_0) - [A \cdot (x - a) - B \cdot (y - b)] \\
&\quad + i\{v(z)) - v(z_0) - [B \cdot (x - a) + A \cdot (y - b)]\}.
\end{aligned}$$

As in Sec. 5, we may divide by $z - z_0$ or $|z - z_0|$ as we wish. If we divide by $z - z_0$, the left side tends to 0 since $A + iB = f'(z_0)$. Thus the right-hand side must go to zero as well. Now if we multiply the right-hand side by $(z - z_0)/|z - z_0|$, it must still go to zero, in both the real and imaginary parts. Since the real part has 0 as a limit, u must be differentiable at z_0. Similarly for the imaginary part, v must be differentiable at z_0 as well. Moreover, applying the same technique we used in Section 5, we can show that the numbers A and B are the partial derivatives of both u and v, by considering either the real or the imaginary part. Thus we have

$$A = (\partial u/\partial x)|_{z=z_0} = (\partial v/\partial y)|_{z=z_0}, \tag{6.1}$$

$$B = (\partial v/\partial x)|_{z=z_0} = -(\partial u/\partial y)|_{z=z_0} : \tag{6.2}$$

the Cauchy-Riemann equations hold if $f'(z)$ exists at z_0.

The proof in the other direction is just as easy. Let's assume that u and v are differentiable at z_0 and the partials of u and v at that point satisfy the Cauchy-Riemann equations. Then we can denote them as

$$u(z) - u(z_0) = A \cdot (x - a) + B \cdot (y - b) + S \tag{6.3}$$

$$v(z) - v(z_0) = C \cdot (x - a) + D \cdot (y - b) + T. \tag{6.4}$$

Again, the technique of Section 5 shows that the constants A, B, C, and D correspond to the partial derivatives. Applying the Cauchy-Riemann equations, we have

$$u(z) - u(z_0) = A \cdot (x - a) + B \cdot (y - b) + S \tag{6.5}$$

$$v(z) - v(z_0) = -B \cdot (x - a) + A \cdot (y - b) + T \tag{6.6}$$

where S and T are the remainder terms. We substitute these into $f = u + iv$:

$$\begin{aligned} f(z) - f(z_0) &= (u(z) + iv(z)) - (u(z_0) + iv(z_0)) \\ &= A \cdot (x - a) + B \cdot (y - b) + S + i[-B \cdot (x - a) + A \cdot (y - b) + T] \\ &= A \cdot (x - a) + B \cdot (y - b) + i[-B \cdot (x - a) + A \cdot (y - b)] + S + iT \\ &= (A + iB) \cdot (z - z_0) + S + iT. \end{aligned}$$

Thus, on dividing by $z - z_0$ or $|z - z_0|$ as appropriate, taking the limit as $z \to z_0$, and recalling (5.12) we have that $f'(z_0) = A + iB$, and thus exists. □

7. Taking Stock

We have shown that if f is analytic, then u, v are differentiable and $u_x = v_y$ and $u_y = -v_x$, such that $\partial Q/\partial x - \partial P/\partial y = 0$ in both applications of Green's theorem. Therefore, we can apply the analytic-conditions version of Green's theorem (Thm. 2) to prove the analytic-conditions version of Cauchy's theorem (Thm. 1). All that remains is the proof for Thm. 2.

8. Final Advance

Editor's Note: At this point, you may want to set down this paper for a while, and perhaps come back to it tomorrow. We have covered a lot of ground, and even I (Josiah) am usually pretty tired by the time I read to this point. But the most beautiful part of the proof is in this section, so I hope you come back!)

For linear functions, that is, functions of the form $f(x, y) = Ax + By + c$, we already have a proof of Green's theorem (1.2), because linear functions have continuous partial derivatives, so the standard proofs of Green's theorem apply. (Alternatively, we can demonstrate Green's theorem directly for linear equations. This is done in Appendix A.)

To prove Green's theorem (Thm. 2) for general functions, we use a technique similar to Goursat's. Let us suppose that Green's theorem is applied to a rectangular region R_0 with boundary γ_0; then we wish to show that

$$\int_{\gamma_0} P dx + Q dy - \iint_{R_0} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx dy = 0 \tag{8.1}$$

Since $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ in the case we need to apply Green's theorem, the double integral is zero, and we need only show that

$$\int_{\gamma_0} P dx + Q dy = 0 \tag{8.2}$$

The approach we use here can be used to prove Green's theorem more generally, but eliminating the double-integral at this point will certainly make the proof shorter.

Suppose this is false, then there must be a Δ_0 such that

$$\left| \int_{\gamma_0} P dx + Q dy \right| > \Delta_0 \tag{8.3}$$

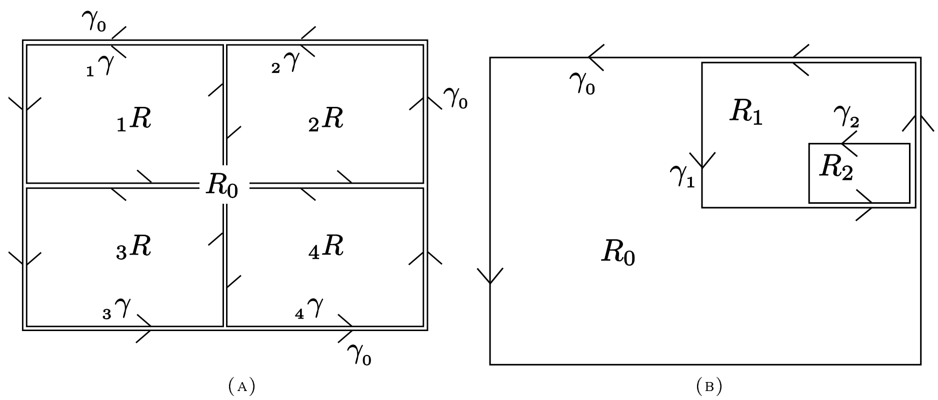

FIGURE 8.1. Dividing the region into quadrants. (a) At each level, the rectangular region is divided into quadrants. If the total contribution from γ_0 to 8.3 is greater than Δ_0, then the contribution from at least one of ${}_1\gamma$, ${}_2\gamma$, ${}_3\gamma$, or ${}_4\gamma$ must be at least $\Delta_1 \geq \Delta_0/4$. (b) We continue this, finding successively smaller regions such that $\Delta_n \geq \Delta_0/4^n$. Here, $R_1 = {}_2R_0$ and $R_2 = {}_4R_1$. Note that there are no margins between the rectangles, these are included in the figure to allow the sub-regions to be distinguished more easily.

We will prove that Δ_0 does not exist by contradiction. Here is Goursat's idea. Suppose we divide R_0 into quadrants ${}_1R$, ${}_2R$, ${}_3R$, and ${}_4R$ as illustrated in Figure 8.1a, and look at the integrals[5]

$$\left| \int_{i\gamma} Pdx + Qdy \right|, \; i = 1, 2, 3, 4 \tag{8.4}$$

We can't have all four of these differences less than $\Delta_0/4$, for if they were, we could add them up and the sum would be less than Δ_0. That means that there must be one smaller rectangle, each side of which is half that of R_0, for which the difference in the two terms is at least $\Delta_0/4$. We shall call this rectangle R_1, its border γ_1, and the difference in the two terms $\Delta_1 \geq \Delta_0/4$.

Now you might be wondering why we can add up the line integrals in (8.4) just like the area integrals. This is a good question. The sum of the four quadrant line integrals add to form the line integral around the main rectangle because, along the interior boundaries, the integrals cancel out. For more details on this, see Appendix B.

Now we repeat this argument with R_1 and divide it into four smaller rectangles (Fig. 8.1b); for one of them, we have that the difference $\Delta_2 \geq \Delta_1/4 \geq \Delta_0/4^2$. We call this R_2, bounded by γ_2. Continuing in this fashion, for each positive integer n we find a rectangle R_n inside $R_{n-1}$with boundary γ_n such that

$$\left| \int_{\gamma_n} Pdx + Qdy \right| = \Delta_n \geq \Delta_0/4^n \tag{8.5}$$

We shall show that for large enough n, no such region exists.

But continuing to suppose that some $\Delta_0 > 0$ does exist (and thus that Cauchy and Green's theorems are false...), such a region could be found for arbitrarily-large n. We

[5]Note that we use pre-scripts like ${}_1R$ to represent the four quadrants, and post-scripts like R_1 to represent levels of nested quadrants which we will introduce shortly.

accept as fact that there must be some point z_0 which is in every one of the nested rectangular regions, i.e. $z_o \in R_i$, $i = 0, 1, 2, \ldots,$. This is itself a theorem from topology, which is discussed in Appendix C, but we hope that it seems obvious enough. Of course, we have stated in the requirements for the proof that P and Q must be differentiable inside R_0 and on γ_0, so they must be differentiable in the neighborhood of z_0, and we have

$$\begin{aligned} P(z) &= P(z_0) + A \cdot (x-a) + B \cdot (y-b) + S(z) \\ Q(z) &= Q(z_0) + C \cdot (x-a) + D \cdot (y-b) + T(z) \end{aligned} \tag{8.6}$$

where A, B, C, and D are the partial derivatives at $z_0 = a + ib$ and S and T satisfy

$$\lim_{z \to z_0} \frac{S(z)}{|z - z_0|} = 0, \qquad \lim_{z \to z_0} \frac{T(z)}{|z - z_0|} = 0 \tag{8.7}$$

At the beginning of this section, we observed that (1.2) is true when P and Q are linear, and so on consulting (8.6), we see that we need only consider the case that $P(z) = S(z)$, $Q(z) = T(z)$.

At this point, you may wish to ask, "But aren't we using Green's Theorem to prove Green's Theorem?" This is a very good question. Yes, we are using a proof of Green's theorem that applies for linear functions. This version can be proved using the standard proofs for Green's theorems found in the textbooks, or by evaluating the integrals directly, as will be done in Appendix A. The proof we are giving here extends Green's theorem to all analytic functions, regardless of whether they have continuous partial derivatives. Of course, all analytic functions have continuous partial derivatives, but as we mentioned in the introduction, the easiest way to prove this is using Cauchy's theorem!

Once we eliminate the linear parts of $P(z)$ and $Q(z)$ at z_0, all we have left to do is to show that there is no $\Delta_0 > 0$ such that

$$\left| \int_{\gamma_n} S dx + T dy \right| = \Delta_n \geq \Delta_0 / 4^n \tag{8.8}$$

Let's focus for a moment on an upper bound for the line integral $\left| \int_{\gamma_n} S dx \right|$. From 8.7 and by the definition of a limit, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all z in the circle $|z - z_0| < \delta$ we are guaranteed that

$$S(z)/|z - z_0| < \epsilon \tag{8.9}$$

Now suppose we select n large enough that the entire region R_n fits inside this circle (the larger circle in Figure 8.2). Now consider the smaller circle $|z - z_0| < \delta_n$, where $\delta_n$is the distance from the center z_n of the rectangle to the edge, . Because this circle lies entirely inside the larger circle, equation (8.9) applies and we can further say that for z in $|z - z_0| < \delta_n$,

$$S(z) < |z - z_0| \epsilon < \delta_n \epsilon. \tag{8.10}$$

This gives us a tight upper bound for the value of $S(z)$ along the line integral of Eq. (8.8). Can we find a similar bound for the path length? Each side of the rectangle is smaller than $2\delta_n$, so $|\gamma_n| < 8\delta_n$. Thus the line integral (8.8) is bounded by

$$\left| \int_{\gamma_n} S dx \right| < (\delta_n \epsilon)(8\delta_n) \tag{8.11}$$

But here's the fun part! We can bound $\delta_n \leq \delta_0 2^{-n}$ where δ_0 is the length of the diagonal of the original rectangle R_0, since δ_n is shorter than the diagonal of R_n. So an n exists such that our bound on the original rectangle is

$$\left| \int_{\gamma_n} S dx \right| < (\epsilon \delta_0 2^{-n})(8\delta_0 2^{-n}) = \frac{\epsilon 8 \delta_0^2}{4^n} \tag{8.12}$$

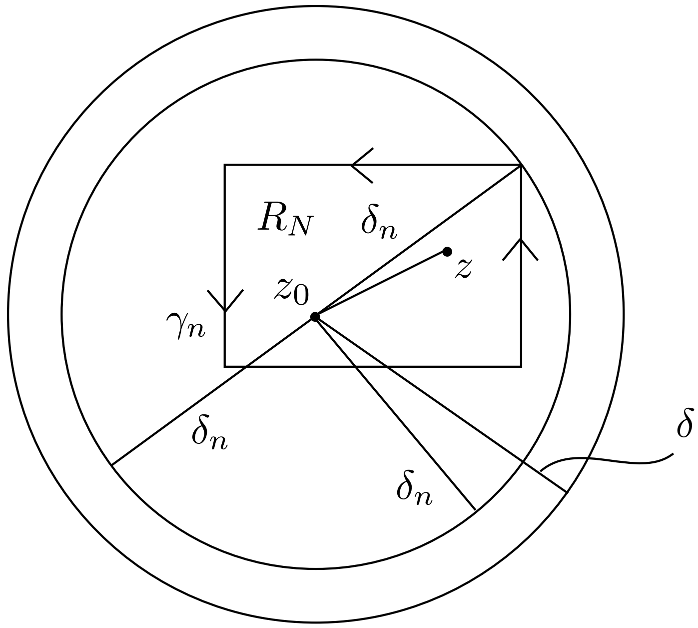

FIGURE 8.2. Limiting circle

where ϵ may be taken as small as we wish. The same bound applies for $T(z)$[6], and we can bound the entire line integral (8.8) as

$$\left|\int_{\gamma_n} S(z)dx + T(z)dy\right| \le \left|\int_{\gamma_n} S(z)dx\right| + \left|\int_{\gamma_n} T(z)dy\right| < 2\frac{\epsilon 8\delta_0^2}{4^n} = \frac{\epsilon 16\delta_0^2}{4^n} \tag{8.13}$$

Thus, for any $\epsilon > 0$, an n exists such that

$$\left|\iint_{\gamma_n} P(z)dx + Q(z)dy - \iint_{R_n}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy\right| = \Delta_n < \frac{\epsilon 16\delta_0^2}{4^n} = eC_0 4^{-n} \tag{8.14}$$

where $C_0 = 16d_0^2$ is fixed with respect to n and ϵ. This contradicts (8.8), which stated that $\Delta_n \ge \Delta_0 4^{-n}$ for all n.[7]

This contradiction proves that our version of Cauchy's theorem holds precisely under the hypothesis that $f(z)$ has a derivative at each point of our rectangle R and its boundary.

Appendix A: Green's Theorem For Linear Functions

In our final advance (Sec. 8), we state that it is easy to prove Green's theorem when P and Q are linear functions and $\partial Q/\partial x - \partial P/\partial y = 0$, that is

$$P(x, y) = A + Bx + Cy \tag{8.15}$$

$$Q(x, y) = D + Ex + Fy \tag{8.16}$$

$$E - C = 0 \tag{8.17}$$

[6]... though admitedly it may be for a different circle δ. But hey, we can always choose the smaller of them right? This is mathematics, not engineering! What's an extra factor of 10 or 100 between friends, when we have an ϵ to cancel it out?

[7]As an example of the contradiction, suppose that $C* = 0.5$, and that we hypothesize that $\Delta_0 = 0.2$. We can show this Δ_0 is impossible by selecting (for example) $\epsilon = \Delta_0/C^*/2 = 0.2$ such that $\Delta_n \le (0.1)(4^{-n})$ for sufficiently large n and $\Delta \ge (0.2)(4^{-n})$ for all n, a contradiction. This same procedure could be used to prove that any $\Delta_0 > 0$ will not work. Thus Δ_0 must be 0.

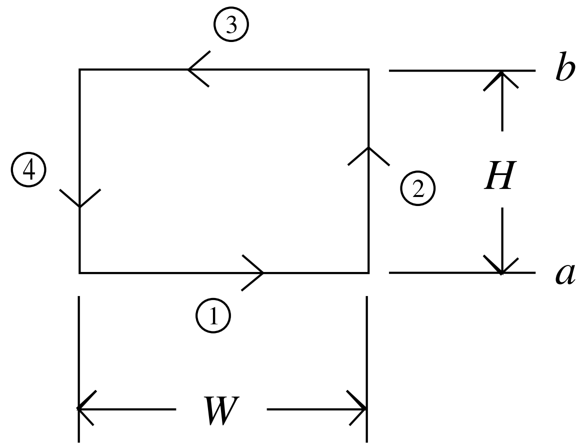

FIGURE 8.3. A very simple line integral for proving Green's Theorem for the linear case.

where A, B, C, D, E, and F are constants. In this case, we can simply use one of the common proof of Green's theorem which only work when P and Q are continuous. But since it is not hard to provide a special proof just for linear functions, we will do that here.

Here, we will consider the case where the γ is a rectangle with height H and width W as shown in Figure 8.3. The parts which only involve the variable of integration will cancel because the outward-bound line integral will be matched by the returning one:

$$\int_\gamma A + Bx\,dx = 0 \tag{8.18}$$

$$\int_\gamma D + Fy\,dy = 0 \tag{8.19}$$

so that

$$\int_\gamma P(x,y)dx - \int_\gamma Q(x,y)dy = \int_\gamma Cy\,dx - \int_\gamma Ex\,dy \tag{8.20}$$

Let's evaluate $\int_\gamma Cy\,dx = \int_{\textcircled{1}} Cy\,dx + \int_{\textcircled{2}} Cy\,dx + \int_{\textcircled{3}} Cy\,dx + \int_{\textcircled{4}} Cy\,dx = \int_{\textcircled{1}} Cy\,dx + \int_{\textcircled{3}} Cy\,dx = CaW - CbW = C(a-b)W = -CHW$. (The integrals along $\textcircled{2}$ and $\textcircled{4}$ are 0 because $dy/dt = 0$ along these curves.) Similarly, $\int_\gamma Ex\,dy = EHW$. Since we have $E - C = 0$, the whole contour integral comes to $\int_\gamma Pdx + Qdy = -CHW + EHW = 0$.

APPENDIX B

For double integrals, it is no surprise that the sum of the integrals in the quadrants is equal to the sum of the integral over the whole rectangle (e.g. Fig. 8.1a). But what about the line integrals?

As an example, let's prove that for Figure 8.4, $\int_{\gamma_{sum}} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$. Let's assume that all of the integrals are with $f(z)dz$ and simply not write this for convenience. From the figure, $\int_{\gamma_1} = \int_{\textcircled{1}} + \int_{\textcircled{2}} + \int_{\textcircled{3}} + \int_{\textcircled{4}}$. Similiarly, $\int_{\gamma_2} = \int_{\textcircled{5}} + \int_{\textcircled{6}} + \int_{\textcircled{7}} + \int_{\textcircled{8}}$. Now since $\int_{\textcircled{8}}$ is along the same line as $\int_{\textcircled{2}}$, but in the opposite direction, we have $\int_{\textcircled{8}} = -\int_{\textcircled{2}}$. So $\int_{\gamma_1} + \int_{\gamma_2} = \int_{\textcircled{1}} + \int_{\textcircled{3}} + \int_{\textcircled{4}} + \int_{\textcircled{5}} + \int_{\textcircled{6}} + \int_{\textcircled{7}} = \int_{\gamma_{sum}}$.

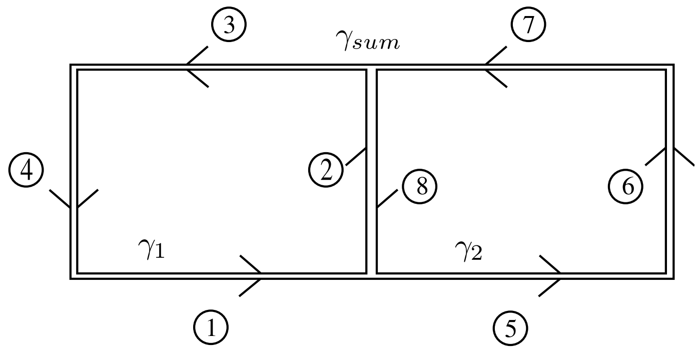

FIGURE 8.4. Adding line integrals

APPENDIX C

Theorem 4. *For all sequences of nested rectangles $R_0, R_1, R_2 \ldots$, such that $R_{i+1} \subset R_i$ for all i, there exists a point z_0 which is within all the rectangles, i.e. $z_0 \in R_i$, $i = 1, 2, 3, \ldots$.*

Proof. This theorem boils down to assuming that the real numbers are continuous. If we consider the x coordinates of the points, saying $z \in R$ means that $x \in [x_i, x_f]$ where $[x_i, x_f]$ is some range of the real line. We start by proving that two points cannot be inside all of the nested rectangles. Suppose the two points z_1 and z_2 have different x coordinates such that $|x_1 - x_2| > \Delta > 0$. (If $x_1 = x_2$, we would use y instead.) Now we can always divide the rectangles smaller and smaller until the width of the rectangle is less than Δ. *So* $|x_1 - x_2| < \Delta$ for all the points inside the rectangle, and clearly both x_1 and x_2 cannot be in the same rectangle. This leads to the conclusion that at most one point can be inside all the rectangles.

But could it be that no point is inside all of the rectangles? This could happen for example if we only allowed x to be a rational number. To say that x must exist is rather like saying the real number line doesn't have any holes. This is closely related to the definition of the real numbers, which we do not have space to discuss here. If you are interested in more information, the Wikipedia article "Construction of the real numbers" looks like a good place to start. □

APPENDIX D

Here we consider standard extensions that make the proof of Cauchy's theorem even more general. Our original proof looked only at rectangles and required the function to be differentiable on the boundary. We shall first relax the condition to allow the function to be merely continuous on the boundary. Then we shall look at how the proof can be extended to non-rectangular regions.

Theorem 5. *Cauchy's theorem also holds if the function is analytic inside γ but only continuous on γ.*

Proof. Not yet included. Has to do with the limiting rectangle growing to fill up the other rectangle. □

Theorem 6. *Cauchy's Theorem also holds for smooth contours.*

Proof. Not yet included. A similar limiting argument. Look at stair-step, and bound error using the continuity of the function: $|f_n(x) - f(x)| < \epsilon/(b-a)$ so the $\int_a^b error \leq \epsilon$. □