Real differentiability and the complex derivative

This is a project consisting of several steps which you can follow and thus get an insight on the relation between the complex derivative and the advanced calculus notion of (real) differentiability

1. Let $f = u + iv$ be complex-valued near $z = z_0$. Then the derivative $f'(z_0)$ exists if

2a. Let $u(x, y)$ be defined in a neighborhood of $z_0 = (x_0, y_0)$. Then u is *differentiable* at z_0 if

where

b. Show that if u is differentiable at z_0 then the partials $u_x,\ u_y$ exist at z_0

c. (¡) Find an example of a function $u(x, y)$ which is differentiable at $(0, 0)$ and for which the partial derivatives exist only at $(0, 0)$. (This has to be a contrived example.)

3a. Let $f'(z_0)$ exist. Show that this means that u and v are differentiable at z_0. (Hint: write out the difference quotient defining f' and consider real and imaginary parts.)

b. Show that u and v satisfy the Cauchy-Riemann equations at z_0. (This was done in class!)

c. Show that the *converse* is true: if u and v are differentiable at z_0 and satisfy the Cauchy-Riemann equations there, then $f'(z_0)$ exists (this was done in class but you should make a good copy for your notes)