Answer Keys for Version 01

1. $$\int \sin^3 x \cos^2 x \, dx$$

$$= \int \sin^2 x \cos^2 x \cdot \sin x \, dx$$

$$= \int (1-\cos^2 x) \cos^2 x \sin x \, dx$$

$$\left(\text{Set} \quad \begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array} \right)$$

$$= \int (1-u^2) u^2 (-du)$$

$$= \int (u^4 - u^2) \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

$$\boxed{\text{Answer} \quad \text{C.} \quad -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C.}$$

2. $$\int_0^{\pi} \sin^2 x \, dx.$$

$$= \int_0^{\pi} \frac{1 - \cos 2x}{2} \, dx.$$

$$= \frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi}$$

$$= \frac{1}{2} \left[(\pi - 0) - (0 - 0) \right] = \frac{1}{2} \pi$$

$\boxed{\text{Answer} \quad A.\ \frac{1}{2}\pi.}$

3. $$\int \frac{\tan^2\theta}{\cos^4\theta}\, d\theta$$

$$= \int \tan^2\theta \sec^4\theta \, d\theta$$

$$= \int \tan^2\theta \sec^2\theta \sec^2\theta \, d\theta$$

$$= \int \tan^2\theta \,(\tan^2\theta + 1) \sec^2\theta \, d\theta$$

$$\left(\text{Set} \quad \begin{aligned} u &= \tan\theta \\ du &= \sec^2\theta \, d\theta \end{aligned} \right)$$

$$= \int u^2 (u^2+1)\, du.$$

$$= \int (u^4 + u^2)\, du.$$

$$= \frac{u^5}{5} + \frac{u^3}{3} + C.$$

$$= \frac{\tan^5\theta}{5} + \frac{\tan^3\theta}{3} + C.$$

$$\boxed{\text{Answer} \quad B. \quad \frac{\tan^5\theta}{5} + \frac{\tan^3\theta}{3} + C}$$

4.
$$\int_0^{\pi/4} \tan^3\theta \sec\theta \, d\theta$$

$$= \int_0^{\pi/4} \tan^2\theta \cdot \tan\theta \sec\theta \, d\theta$$

$$= \int_0^{\pi/4} (\sec^2\theta - 1) \cdot \tan\theta \sec\theta \, d\theta$$

$$\left(\begin{array}{lcl} \text{Set} & u = & \sec\theta \\ & du = & \tan\theta \sec\theta \, d\theta \\ & \theta & u \\ & \pi/4 & \sqrt{2} \\ & 0 & 1 \end{array} \right)$$

$$= \int_0^{\sqrt{2}} (u^2 - 1) \, du$$

$$= \left[\frac{u^3}{3} - u \right]_1^{\sqrt{2}}$$

$$= \left[\left(\frac{2\sqrt{2}}{3} - \sqrt{2} \right) - \left(\frac{1}{3} - 1 \right) \right] = -\frac{\sqrt{2}}{3} + \frac{2}{3}$$

$$\boxed{\text{Answer} \quad A. \ \frac{2-\sqrt{2}}{3}}$$

5. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$

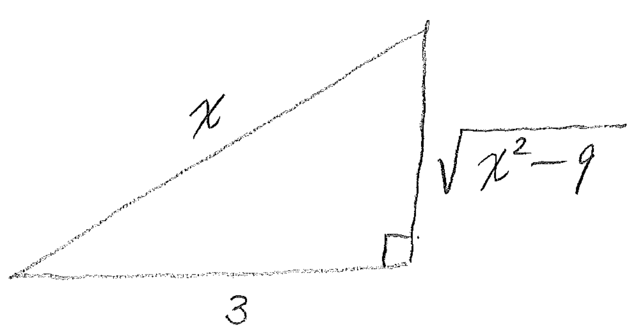
$$\left(\begin{aligned} \text{Set} \quad x &= 3\sec\theta \\ dx &= 3\sec\theta\tan\theta\,d\theta \\ \sqrt{x^2-9} &= 3\tan\theta \end{aligned}\right)$$

$$= \int \frac{1}{(3\sec\theta)^2\, 3\tan\theta}\cdot 3\sec\theta\tan\theta\,d\theta$$

$$= \frac{1}{9}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{9}\int \cos\theta\,d\theta$$

$$= \frac{1}{9}\sin\theta + C.$$

Consider



$$x = 3\sec\theta$$

$$\frac{x}{3} = \sec\theta$$

$$\sin\theta = \frac{\sqrt{x^2-9}}{x}$$

$$= \frac{1}{9}\cdot\frac{\sqrt{x^2-9}}{x} + C.$$

$$\boxed{\text{Answer} \quad \text{D.} \quad \frac{\sqrt{x^2-9}}{9x} + C.}$$

6. $$\int \frac{1}{\sqrt{x^2-6x+13}}\,dx$$

$$= \int \frac{1}{\sqrt{(x-3)^2+4}}\,dx$$

$$\left(\begin{aligned} \text{Set} \quad u &= x-3 \\ du &= dx \end{aligned} \right)$$

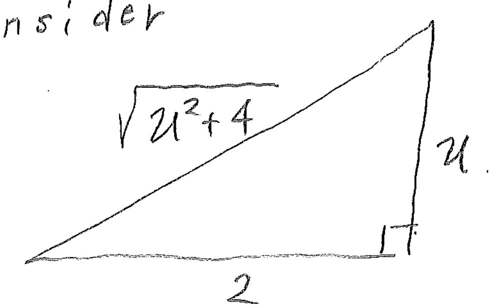
$$= \int \frac{1}{\sqrt{u^2+4}}\,du$$

$$\left(\begin{aligned} \text{Set} \quad u &= 2\tan\theta \\ du &= 2\sec^2\theta\,d\theta \\ \sqrt{u^2+4} &= 2\sec\theta \end{aligned} \right)$$

$$= \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta = \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C'$$

(Consider



$$u = 2\tan\theta$$
$$\frac{u}{2} = \tan\theta$$
$$\sec\theta = \frac{\sqrt{u^2+4}}{2}$$)

$$= \ln \left| \frac{\sqrt{u^2+4}}{2} + \frac{u}{2} \right| + C'$$

$$= \ln \left(\frac{1}{2} \left| \sqrt{u^2+4} + u \right| \right) + C'$$

$$= \ln \left| \sqrt{u^2+4} + u \right| + C$$

$$\left(\text{where } C = C' + \ln \frac{1}{2} \right)$$

$$= \ln \left| \sqrt{(x-3)^2+4} + (x-3) \right| + C$$

$$\boxed{\text{Answer A. } \ln \left| \sqrt{x^2-6x+13} + x - 3 \right| + C}$$

7. $$\frac{x^2}{x^2-4} = \frac{(x^2-4)+4}{x^2-4}$$

$$= 1 + \frac{4}{x^2-4}$$

$$\frac{4}{x^2-4} = \frac{4}{(x-2)(x+2)}$$

$$= \frac{A}{x-2} + \frac{B}{x+2}.$$

$$\left(\begin{array}{lll} & 4 = A(x+2) + B(x-2) & \\ x = 2. & 4 = A \cdot 4 & A = 1. \\ x = -2 & 4 = B \cdot (-4) & B = -1. \end{array} \right)$$

$$= \frac{1}{x-2} + \frac{(-1)}{x+2}$$

Therefore, we have

$$\int \frac{x^2}{x^2-4}\,dx = \int \left\{ 1 + \frac{1}{x-2} + \frac{(-1)}{x+2} \right\} dx.$$

$$= x + \ln|x-2| - \ln|x+2| + C.$$

$$\boxed{\text{Answer C : } x + \ln\left|\frac{x-2}{x+2}\right| + C.}$$

8.

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}.$$

$$10 = A(x^2+9) + (Bx+C)(x-1)$$

$$= (A+B)x^2 + (-B+C)x + 9A-C.$$

$$\begin{cases} A+B=0 \\ -B+C=0 \\ 9A-C=10 \end{cases} \qquad \begin{array}{l} A=1 \\ B=-1 \\ C=-1 \end{array}$$

$$= \frac{1}{x-1} + \frac{(-1)x+(-1)}{x^2+9}.$$

$$= \frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9}.$$

Therefore, we have

$$\int \frac{10}{(x-1)(x^2+9)}\,dx$$

$$= \int \left\{ \frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9} \right\} dx.$$

$$= \int \frac{1}{x-1}\,dx - \int \frac{x}{x^2+9}\,dx - \int \frac{1}{x^2+9}\,dx$$

$$= \ln|x-1| - \frac{1}{2}\ln|x^2+9| - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

since

$$\int \frac{x}{x^2+9}\,dx = \int \frac{\frac{1}{2}\,du}{u}$$

$$\left(\text{Set} \quad \begin{aligned} u &= x^2+9 \\ du &= 2x\,dx \end{aligned} \right)$$

$$= \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C_1$$

$$= \frac{1}{2}\ln|x^2+9| + C_1$$

$$\int \frac{1}{x^2+9}\,dx = \int \frac{1}{9\left\{\left(\frac{x}{3}\right)^2+1\right\}}\,dx$$

$$\left(\text{Set} \quad \begin{aligned} v &= \frac{x}{3} \\ dv &= \frac{1}{3}\,dx \end{aligned} \right)$$

$$= \frac{1}{9}\int \frac{3\,dv}{v^2+1} = \frac{1}{3}\int \frac{dv}{v^2+1}$$

$$= \frac{1}{3}\tan^{-1} v + C_2$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C_2.$$

Answer

B. $\ln|x-1| - \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\tan^{-1}\frac{x}{3} + C.$

9. $$\int_9^{16} \frac{\sqrt{x}}{x-4}\,dx.$$

$$\left(\begin{array}{l} \text{Set} \quad u = \sqrt{x} \\ \qquad du = \frac{1}{2\sqrt{x}}\,dx. \\ \qquad 2u\,du = 2\sqrt{x}\,du = dx. \\ \begin{array}{ll} x & u = \sqrt{x} \\ 16 & 4. \\ 9 & 3 \end{array} \end{array}\right)$$

$$= \int_3^4 \frac{u}{u^2-4}\,2u\,du.$$

$$= 2\int_3^4 \frac{u^2}{u^2-4}\,du. \quad (\text{See } \#7)$$

$$= 2\left[\, u + \ln\left|\frac{u-2}{u+2}\right| \,\right]_3^4.$$

$$= 2\left[\left(4+\ln\frac{2}{6}\right) - \left(3+\ln\frac{1}{5}\right)\right]$$

$$= 2\left[\,1 + \ln\frac{2}{6}\cdot 5\,\right] = 2\left(1+\ln\frac{5}{3}\right).$$

$$\boxed{\text{Answer} \quad \text{E.} \quad 2\left(1+\ln\frac{5}{3}\right)}$$

10.

$$\int_0^{\pi} x^2 \sin x \, dx$$

$\approx$ by Simpson's rule.

$$\frac{1}{3}\left\{ f(0) + 4f\left(\frac{\pi}{4}\right) + 2f\left(\frac{\pi}{2}\right) + 4f\left(\frac{3\pi}{4}\right) + f(\pi) \right\} \cdot \frac{\pi}{4}$$

Answer

C $\frac{\pi}{12}\left\{ f(0) + 4f\left(\frac{\pi}{4}\right) + 2f\left(\frac{\pi}{2}\right) + 4f\left(\frac{3\pi}{4}\right) + f(\pi) \right\}$

11.

(i) $$\int_1^\infty \frac{1}{(2x+1)^3}\,dx$$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{(2x+1)^3}\,dx$$

$$\left(\begin{array}{cc} \text{Set} & u = 2x+1 \\ & du = 2\,dx \\ x & u \\ t & 2t+1 \\ 1 & 3 \end{array} \right)$$

$$= \lim_{t\to\infty} \int_3^{2t+1} \frac{1}{u^3}\cdot\frac{1}{2}\,du$$

$$= \lim_{t\to\infty} \frac{1}{2}\left[-\frac{1}{2}\cdot\frac{1}{u^2}\right]_3^{2t+1}$$

$$= \lim_{t\to\infty} -\frac{1}{4}\left[\frac{1}{(2t+1)^2} - \frac{1}{3^2}\right]$$

$$= -\frac{1}{4}\left(0 - \frac{1}{3^2}\right) = \frac{1}{36}$$

convergent.

(ii) $\displaystyle\int_0^4 \frac{\ln x}{x}\,dx$

$$= \lim_{t \to 0^+} \int_t^4 \frac{\ln x}{x}\,dx$$

$$\left(\begin{array}{l} \text{Set } u = \ln x \\ du = \frac{1}{x}\,dx. \\ \begin{array}{cc} x & u \\ 4 & \ln 4. \\ t & \ln t \end{array} \end{array} \right)$$

$$= \lim_{t \to 0^+} \int_{\ln t}^{\ln 4} u\,du. = \lim_{t \to 0^+} \left[\frac{u^2}{2}\right]_{\ln t}^{\ln 4}.$$

$$= \lim_{t \to 0^+} \left[\frac{(\ln 4)^2}{2} - \frac{(\ln t)^2}{2}\right] = -\infty$$

since $\displaystyle\lim_{t \to 0^+} \ln(t) = -\infty$

divergent.

(iii) $\displaystyle\int_2^3 \frac{1}{\sqrt{3-x}}\,dx$

$$= \lim_{t \to 3^-} \int_2^t \frac{1}{\sqrt{3-x}}\,dx$$

$$\left(\begin{array}{cc} \text{Set} & u = 3 - x \\ & du = -dx \\ x & u \\ t & 3-t \\ 2 & 1 \end{array}\right)$$

$$= \lim_{t \to 3^-} \int_1^{3-t} \frac{1}{\sqrt{u}}\,(-du)$$

$$= -\lim_{t \to 3^-} \int_1^{3-t} \frac{1}{\sqrt{u}}\,du$$

$$= -\lim_{t \to 3^-} \left[\frac{2}{3} u^{\frac{3}{2}}\right]_1^{3-t}$$

$$= -\lim_{t \to 3^-} \left[\frac{2}{3}(3-t)^{\frac{3}{2}} - \frac{2}{3} 1^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}$$

convergent.

Therefore, we conclude that only (ii) is divergent.

$$\boxed{\text{Answer} \quad \text{B. (ii) only}}$$

12.

$$y = f(x) = \ln(\cos x)$$

$$0 \leqq x \leqq \pi/3$$

$$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$$

The length of the curve is given by

$$L = \int_0^{\pi/3} \sqrt{1 + \{f'(x)\}^2}\, dx$$

$$= \int_0^{\pi/3} \sqrt{1 + (-\tan x)^2}\, dx$$

$$= \int_0^{\pi/3} \sqrt{\sec^2 x}\, dx$$

$$= \int_0^{\pi/3} \sec x\, dx$$

$$= \Big[\ln | \sec x + \tan x | \Big]_0^{\pi/3}$$

$$= [\ln(2 + \sqrt{3}) - \ln(1 + 0)]$$

$$= \ln(2 + \sqrt{3})$$

$$\boxed{\text{Answer} \quad C. \quad \ln(2 + \sqrt{3})}$$

13.

The integral for the area of the surface obtained by rotating the curve $y = f(x)$ $a \leq x \leq b$ about the y-axis is given by

$$A_y = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx ,$$

while the one for the area of the surface obtained by rotating the same curve $y = f(x)$ about the x-axis is given by

$$A_x = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx .$$

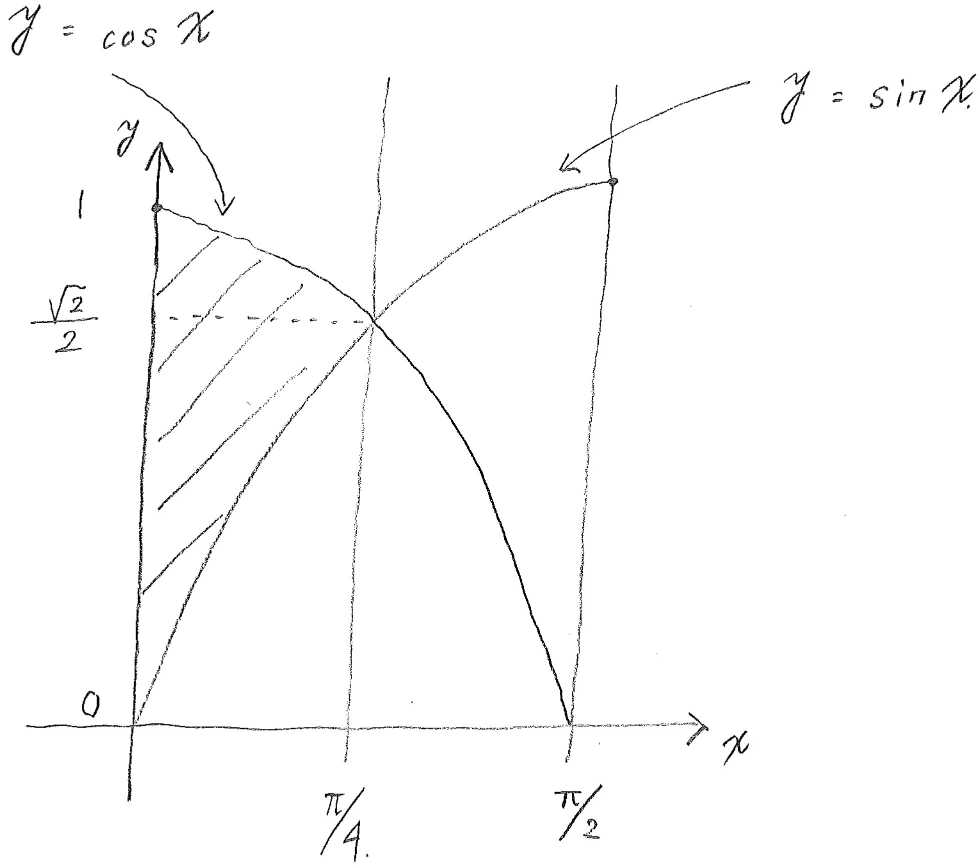
Therefore, the integral for the area of the surface obtained by rotating the curve $y = \tan^{-1} x$, $0 \leq x \leq 1$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

is given by

$$\boxed{\text{Answer} \quad D: \int_0^1 2\pi x \sqrt{1 + \frac{1}{(1+x^2)^2}}\, dx}$$

14.



We compute the area of the region

$$A = \int_0^{\pi/4} (\cos x - \sin x)\, dx$$

$$= \left[\sin x + \cos x \right]_0^{\pi/4}$$

$$= \left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1) \right] = \sqrt{2} - 1.$$

We compute the moment about the x-axis with ρ representing the density

$$M_x = \rho \cdot \frac{1}{2} \int_0^{\pi/4} (\cos^2 x - \sin^2 x)\, dx$$

$$= \rho \cdot \frac{1}{2} \int_0^{\pi/4} \cos 2x \, dx$$

$$= \rho \cdot \frac{1}{2} \left[\frac{1}{2} \sin 2x \right]_0^{\pi/4}$$

$$= \rho \cdot \frac{1}{4}$$

Therefore, we compute

$$\bar{y} = \frac{M_x}{\rho \cdot A} = \frac{\rho \cdot \frac{1}{4}}{\rho(\sqrt{2}-1)} = \frac{1}{4(\sqrt{2}-1)}$$

$$\boxed{\text{Answer} \quad D. \ \frac{1}{4(\sqrt{2}-1)}}$$