Answer Keys for Version 02.

1. $$\int \sin^3 x \, dx$$

$$= \int \sin^2 x \cdot \sin x \, dx$$

$$= \int (1-\cos^2 x) \cdot \sin x \, dx$$

$$\left(\begin{array}{l} \text{Set } u = \cos x. \\ \phantom{\text{Set }} du = -\sin x \, dx. \end{array} \right)$$

$$= \int (1-u^2)(-du)$$

$$= \int (u^2-1)\, du.$$

$$= \frac{u^3}{3} - u + C$$

$$= \frac{\cos^3 x}{3} - \cos x + C.$$

$$\boxed{\text{Answer} \quad B. \quad \frac{\cos^3 x}{3} - \cos x + C}$$

2.

$$\int_0^{\pi} \cos^2 x \, dx$$

$$= \int_0^{\pi} \left(\frac{1 + \cos 2x}{2} \right) dx$$

$$= \frac{1}{2} \int_0^{\pi} (1 + \cos 2x) \, dx$$

$$= \frac{1}{2} \left[x + \frac{1}{2} \sin 2x \right]_0^{\pi}$$

$$= \frac{1}{2} \left[(\pi + 0) - (0 + 0) \right] = \frac{1}{2} \pi.$$

$\boxed{\text{Answer} \quad \text{D.} \ \frac{1}{2} \pi.}$

3.

$$\int \frac{\tan^2\theta}{\cos^6\theta}\, d\theta$$

$$= \int \tan^2\theta \sec^6\theta\, d\theta$$

$$= \int \tan^2\theta \sec^4\theta \sec^2\theta\, d\theta$$

$$= \int \tan^2\theta \left(\tan^2\theta + 1\right)^2 \sec^2\theta\, d\theta$$

$$\left(\begin{array}{lcl}\text{Set} \quad u & = & \tan\theta \\ du & = & \sec^2\theta\, d\theta\end{array}\right)$$

$$= \int u^2 (u^2 + 1)^2\, du.$$

$$= \int (u^6 + 2u^4 + u^2)\, du.$$

$$= \frac{u^7}{7} + 2\frac{u^5}{5} + \frac{u^3}{3} + C.$$

$$= \frac{\tan^7\theta}{7} + \frac{2}{5}\tan^5\theta + \frac{\tan^3\theta}{3} + C.$$

$$\boxed{\text{Answer} \quad A. \quad \frac{\tan^7\theta}{7} + \frac{2}{5}\tan^5\theta + \frac{\tan^3\theta}{3} + C.}$$

4. $\displaystyle\int_0^{\pi/4} \tan^3\theta \sec\theta \, d\theta$

$$= \int_0^{\pi/4} \tan^2\theta \tan\theta \sec\theta \, d\theta$$

$$= \int_0^{\pi/4} (\sec^2\theta - 1) \tan\theta \sec\theta \, d\theta$$

$$\left(\begin{array}{ll} \text{Set} & u = \sec\theta \\ & du = \tan\theta \sec\theta \, d\theta \\ & \\ \theta & \sec\theta \\ \pi/4 & \sqrt{2}. \\ 0 & 1. \end{array} \right)$$

$$= \int_1^{\sqrt{2}} (u^2 - 1) \, du.$$

$$= \left[\frac{u^3}{3} - u \right]_1^{\sqrt{2}} = \left[\left(\frac{2}{3}\sqrt{2} - \sqrt{2} \right) - \left(\frac{1}{3} - 1 \right) \right]$$

$$= \frac{2 - \sqrt{2}}{3}$$

$$\boxed{\text{Answer} \quad \text{A.} \quad \frac{2 - \sqrt{2}}{3}}$$

5.

$$\int \frac{1}{x^2\sqrt{x^2-16}}\,dx$$

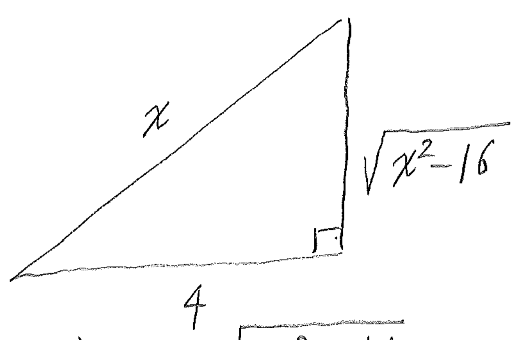
$$\left(\begin{aligned} \text{Set} \quad x &= 4\sec\theta \\ dx &= 4\sec\theta\tan\theta\,d\theta \\ \sqrt{x^2-16} &= 4\tan\theta \end{aligned}\right)$$

$$= \int \frac{4\sec\theta\tan\theta}{(4\sec\theta)^2\,4\tan\theta}\,d\theta$$

$$= \frac{1}{16}\int\frac{1}{\sec\theta}\,d\theta = \frac{1}{16}\int\cos\theta\,d\theta$$

$$= \frac{1}{16}\sin\theta + C$$

Consider



$$x = 4\sec\theta$$

$$\frac{x}{4} = \sec\theta$$

$$\sin\theta = \frac{\sqrt{x^2-16}}{x}.$$

$$= \frac{1}{16}\cdot\frac{\sqrt{x^2-16}}{x} + C.$$

$$\boxed{\text{Answer B. } \frac{\sqrt{x^2-16}}{16x} + C.}$$

6.
$$\int \frac{1}{\sqrt{x^2-4x+13}}\,dx$$

$$= \int \frac{1}{\sqrt{(x-2)^2+9}}\,dx$$

$$\left(\begin{array}{l} \text{Set } u = x-2 \\ \qquad du = dx \end{array} \right)$$

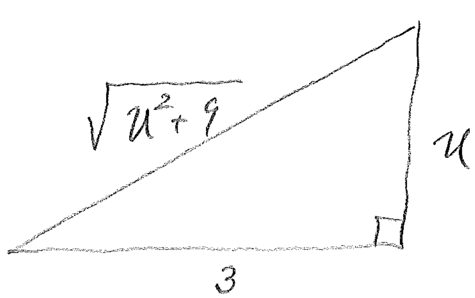
$$= \int \frac{1}{\sqrt{u^2+9}}\,du.$$

$$\left(\begin{array}{rl} \text{Set } \quad u &= 3\tan\theta \\ du &= 3\sec^2\theta\,d\theta \\ \sqrt{u^2+9} &= 3\sec\theta. \end{array} \right)$$

$$= \int \frac{3\sec^2\theta}{3\sec\theta}\,d\theta = \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C'$$

(Consider



$$u = 3\tan\theta$$

$$\frac{u}{3} = \tan\theta$$

$$\sec\theta = \frac{\sqrt{u^2+9}}{3}$$
)

$$= \ln \left| \frac{\sqrt{u^2+9}}{3} + \frac{u}{3} \right| + c'$$

$$= \ln \left(\frac{1}{3} \left| \sqrt{u^2+9} + u \right| \right) + c'$$

$$= \ln \left| \sqrt{u^2+9} + u \right| + C$$

$$\left(\text{where } C = \ln \frac{1}{3} + c'\right)$$

$$= \ln \left| \sqrt{(x-2)^2+9} + (x-2) \right| + C.$$

$$\boxed{\text{Answer} \quad A.\ \ln \left| \sqrt{x^2-4x+13} + x - 2 \right| + C.}$$

7. $$\frac{x^2}{x^2-9} = \frac{(x^2-9)+9}{x^2-9}$$
$$= 1 + \frac{9}{x^2-9}.$$

$$\frac{9}{x^2-9} = \frac{9}{(x-3)(x+3)}$$
$$= \frac{A}{x-3} + \frac{B}{x+3}.$$

$$\left(\begin{array}{llll} & 9 = A(x+3) + B(x-3) & & \\ x=3 & 9 = A\cdot 6 & & A = \frac{9}{6} = \frac{3}{2} \\ x=-3 & 9 = B(-6) & & B = -\frac{9}{6} = -\frac{3}{2}. \end{array}\right)$$

$$= \frac{3}{2}\cdot\frac{1}{x-3} + \left(-\frac{3}{2}\right)\cdot\frac{1}{x+3}$$

Therefore, we have

$$\int \frac{x^2}{x^2-9}\,dx = \int\left\{1 + \frac{3}{2}\cdot\frac{1}{x-3} - \frac{3}{2}\cdot\frac{1}{x+3}\right\}dx$$
$$= x + \frac{3}{2}\ln|x-3| - \frac{3}{2}\ln|x+3| + C.$$

$$\boxed{\text{Answer B. } \quad x + \frac{3}{2}\ln\left|\frac{x-3}{x+3}\right| + C.}$$

8. $$\frac{13}{(x-3)(x^2+4)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+4}$$

$$\left(\begin{aligned} 13 &= A(x^2+4) + (Bx+C)(x-3) \\ &= (A+B)x^2 + (-3B+C)x + 4A - 3C \\ & \begin{cases} A + B = 0 \\ -3B + C = 0 \\ 4A - 3C = 13 \end{cases} \qquad \begin{aligned} A &= 1 \\ B &= -1 \\ C &= -3 \end{aligned} \end{aligned} \right)$$

$$= \frac{1}{x-3} + \frac{(-1)x + (-3)}{x^2+4}$$

$$= \frac{1}{x-3} - \frac{x}{x^2+4} - \frac{3}{x^2+4}$$

Therefore, we have

$$\int \frac{13}{(x-3)(x^2+4)}\,dx$$

$$= \int \left\{ \frac{1}{x-3} - \frac{x}{x^2+4} - \frac{3}{x^2+4} \right\} dx$$

$$= \int \frac{1}{x-3}\,dx - \int \frac{x}{x^2+4}\,dx - \int \frac{3}{x^2+4}\,dx$$

$$= \ln|x-3| - \frac{1}{2}\ln|x^2+4| - \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

since

$$\int \frac{x}{x^2+4}\,dx = \int \frac{\frac{1}{2}\,du}{u}.$$

$$\left(\begin{array}{l} \text{Set } u = x^2+4. \\ \quad du = 2x\,dx \end{array} \right)$$

$$= \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C_1$$

$$= \frac{1}{2}\ln|x^2+4| + C_1$$

$$\int \frac{1}{x^2+4}\,dx = \int \frac{1}{4\left\{\left(\frac{x}{2}\right)^2+1\right\}}\,dx$$

$$\left(\begin{array}{l} \text{Set } v = \frac{x}{2} \\ \quad dv = \frac{1}{2}\,dx \end{array} \right)$$

$$= \frac{1}{4}\int \frac{2\,dv}{v^2+1.} = \frac{1}{2}\int \frac{dv}{v^2+1}$$

$$= \frac{1}{2}\tan^{-1}v + C_2$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C_2$$

Answer

E. $\ln|x-3| - \frac{1}{2}\ln(x^2+4) - \frac{3}{2}\tan^{-1}\frac{x}{2} + C$

9. $$\int_9^{16} \frac{\sqrt{x}}{x-4}\, dx$$

Set $u = \sqrt{x}$

$$du = \frac{1}{2\sqrt{x}}\, dx$$

$$2u\, du = 2\sqrt{x}\, du = dx$$

x	$u = \sqrt{x}$
16	4
9	3.

$$= \int_3^4 \frac{u^2}{u^2-4}\, 2u\, du$$

$$= 2\int_3^4 \frac{u^2}{u^2-4}\, du \quad \text{(similer to \#7)}$$

$$= 2\left[u + \ln\left|\frac{u-2}{u+2}\right| \right]_3^4$$

$$= 2\left[\left(4 + \ln\frac{2}{6}\right) - \left(3 + \ln\frac{1}{5}\right) \right]$$

$$= 2\left[1 + \ln\frac{2}{6}\cdot 5 \right] = 2\left(1 + \ln\frac{5}{3}\right)$$

$$\boxed{\text{Answer C. } 2\left(1 + \ln\frac{5}{3}\right)}$$

10.

$$\int_0^{\pi} x^2 \sin x \, dx$$

$\approx$ by Simpson's rule

$$\frac{1}{3}\left\{ f(0) + 4f\left(\frac{\pi}{4}\right) + 2f\left(\frac{\pi}{2}\right) + 4f\left(\frac{3\pi}{4}\right) + f(\pi) \right\} \cdot \frac{\pi}{4}.$$

Answer

E. $\frac{\pi}{12}\left\{ f(0) + 4f\left(\frac{\pi}{4}\right) + 2f\left(\frac{\pi}{2}\right) + 4\left(\frac{3\pi}{4}\right) + f(\pi) \right\}$.

11.

(i) $$\int_0^1 \frac{1}{2-3x}\,dx$$

$$= \int_0^{2/3} \frac{1}{2-3x}\,dx + \int_{2/3}^{1} \frac{1}{2-3x}\,dx$$

$$\int_0^{2/3} \frac{1}{2-3x}\,dx = \lim_{t \to \frac{2}{3}^-} \int_0^t \frac{1}{2-3x}\,dx$$

$$= \lim_{t \to \frac{2}{3}^-} \left[-\frac{1}{3} \ln |2-3x| \right]_0^t$$

$$= \lim_{t \to \frac{2}{3}^-} \left[\left(-\frac{1}{3} \ln |2-3t|\right) - \left(-\frac{1}{3} \ln 2\right) \right]$$

$$= +\infty$$ divergent

Therefore, we conclude

$$\int_0^1 \frac{1}{2-3x}\,dx$$ divergent.

(ii) $$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{t \to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{t \to 0^+} \left[\frac{2}{3} x^{\frac{3}{2}} \right]_t^1$$

$$= \lim_{t \to 0^+} \frac{2}{3} \left[1^{\frac{3}{2}} - t^{\frac{3}{2}} \right] = 1.$$

convergent

(iii) $$\int_{2\pi}^{\infty} \sin\theta \, d\theta$$

$$= \lim_{t \to \infty} \int_{2\pi}^{t} \sin\theta \, d\theta$$

$$= \lim_{t \to \infty} \left[-\cos\theta \right]_{2\pi}^{t}$$

$$= \lim_{t \to \infty} - \left[\cos t - \cos 2\pi \right]$$

This limit does not exist, since $\cos t$ oscillates as t goes to infinity.

divergent.

$\boxed{\text{Answer D. (i) and (iii)}}$

12. $y = f(x) = \ln(\cos x)$

$$0 \leqq x \leqq \pi/3$$

$$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$$

The length of the curve is given by

$$L = \int_0^{\pi/3} \sqrt{1 + \{f'(x)\}^2}\, dx$$

$$= \int_0^{\pi/3} \sqrt{1 + (-\tan x)^2}\, dx$$

$$= \int_0^{\pi/3} \sqrt{\sec^2 x}\, dx$$

$$= \int_0^{\pi/3} \sec x \, dx$$

$$= \Big[\ln | \sec x + \tan x | \Big]_0^{\pi/2}$$

$$= [\ln(2+\sqrt{3}) - \ln(1+0)]$$

$$= \ln(2+\sqrt{3})$$

$$\boxed{\text{Answer} \quad E. \quad \ln(2+\sqrt{3})}$$

13. The integral for the area of the surface obtained by rotating the curve $y = f(x)$ $a \leq x \leq b$ about the y-axis is given by

$$A_y = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

while the one for the area of the surface obtained by rotating the same curve $y = f(x)$ about the x-axis is given by

$$A_x = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx.$$

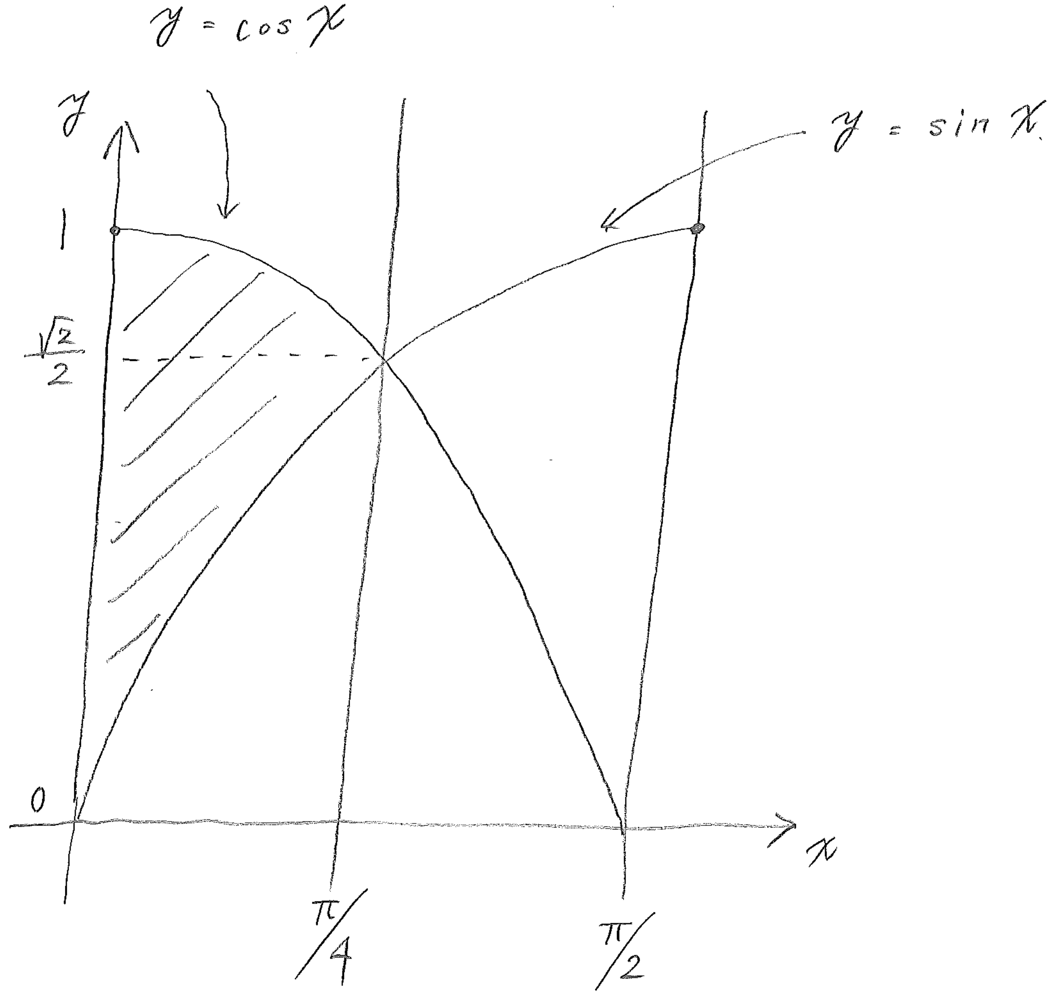
Therefore, the integral for the area of the surface obtained by rotating the curve $y = \sin^{-1} x$, $0 \leq x \leq 1$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

is given by

$$\boxed{\text{Answer} \quad D. \quad \int_0^1 2\pi x \sqrt{1 + \frac{1}{1-x^2}} \cdot dx}$$

14.



We compute the area of the region

$$A = \int_0^{\pi/4} (\cos x - \sin x)\, dx$$

$$= [\sin x + \cos x]_0^{\pi/4}$$

$$= \left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1)\right] = \sqrt{2} - 1$$

We compute the moment about the x-axis with ρ representing the density

$$M_x = \rho \cdot \frac{1}{2} \int_0^{\pi/4} (\cos^2 x - \sin^2 x)\, dx$$

$$= \rho \cdot \frac{1}{2} \int_0^{\pi/4} \cos 2x \, dx$$

$$= \rho \cdot \frac{1}{2} \left[\frac{1}{2} \sin 2x \right]_0^{\pi/4}$$

$$= \rho \cdot \frac{1}{4}$$

Therefore, we compute

$$\bar{y} = \frac{M_x}{\rho \cdot A} = \frac{\rho \cdot \frac{1}{4}}{\rho(\sqrt{2}-1)} = \frac{1}{4(\sqrt{2}-1)}$$

$$\boxed{\text{Answer} \quad B, \quad \frac{1}{4(\sqrt{2}-1)}}$$