Answer Keys for Version 01

1. (i) $\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right)$

$= \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{1/n}$

$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$

Therefore, the sequence $\left\{n \sin\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$ is convergent.

(ii) $\lim_{n\to\infty} \cos(n\pi)$ does not exist.
Therefore, the sequence $\{\cos(n\pi)\}_{n=1}^{\infty}$ is divergent.

(iii) $\lim_{n\to\infty} \frac{n-1}{n} = 1.$

Therefore, the sequence $\left\{\frac{n-1}{n}\right\}_{n=2}^{\infty}$ is convergent.

$\boxed{\text{Answer E. (i) and (iii) only}}$

2.

(i) Set $a_n = \ln\left(\frac{n^2+1}{2n^2+1}\right)$.

Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right) = \ln\frac{1}{2} \neq 0$$

Therefore, by Test for divergence, we conclude that the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right) \text{ is divergent.}$$

(ii) Set $a_n = \arctan(n)$.

Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan(n) = \frac{\pi}{2} \neq 0.$$

Therefore, by Test for divergence, we conclude that the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \arctan(n) \text{ is divergent.}$$

(iii) Set $a_n = \frac{1+3^n}{2^n}$

Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1+3^n}{2^n} = \infty \neq 0.$$

Therefore, by Test for divergence, we conclude that the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1+3^n}{2^n} \text{ is divergent.}$$

$\boxed{\text{Answer A. All.}}$

3. Set

$$a_n = \frac{1}{(n^3+n)^p}$$

$$b_n = \frac{1}{n^{3p}}.$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(n^3+n)^p}{1/n^{3p}}$$

$$= \lim_{n\to\infty} \left(\frac{n^3}{n^3+n}\right)^p = 1.$$

Therefore, by Limit Comparison Test, the two series $\sum a_n$ and $\sum b_n$ share the same destiny.

On the other hand, the condition for the series $\sum b_n = \sum \frac{1}{n^{3p}}$ to converge is $1 < 3p$, i.e. $\frac{1}{3} < p$.

Therefore, we conclude that the condition on the value of p for which the series $\sum a_n$ converges is the same, i.e. $\frac{1}{3} < p$.

$\boxed{\text{Answer} \quad E \quad \frac{1}{3} < p.}$

4. A geometric series with the initial term a and the ratio r $(-1 < r < 1)$ converges to $\dfrac{a}{1-r}$.

Therefore, we evaluate

$$\sum_{n=1}^{\infty}\left[\left(\frac{4}{5}\right)^{n-1}-\left(\frac{3}{10}\right)^{n}\right] = \sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^{n-1}-\sum_{n=1}^{\infty}\left(\frac{3}{10}\right)^{n}$$

$$= \frac{1}{1-\frac{4}{5}} - \frac{3/10}{1-\frac{3}{10}} = 5 - \frac{3}{7} = \frac{32}{7}.$$

$$\boxed{\text{Answer D. } \frac{32}{7}.}$$

5.

A. Set $b_n = \frac{1}{3n^2}$.

Then actually we have.

$0 \leqq b_n \leqq a_n$, and NOT $0 \leqq a_n \leqq b_n$.

Therefore, statement A is wrong.

B. Set $b_n = \frac{1}{3n^2}$.

Then

$0 \leqq b_n \leqq a_n$, but $\sum b_n$ is convergent.

Therefore, statement B is wrong.

C. Set $b_n = \frac{1}{3n^2}$.

Then

$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$ and $\sum b_n$ is convergent.

Therefore, by Limit Comparison Test,

$\sum a_n$ is convergent.

Therefore, statement C is correct.

D. Set $b_n = \frac{1}{3n^2}$

Then

$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$ but $\sum b_n$ is NOT divergent.

Therefore, statement D is wrong.

E. We compute

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+1}{3n^4-1} = 0$$

Therefore, statement E is wrong.

$\boxed{\text{Answer C.}}$

6.

A. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \lim_{\theta\to 0} \sin\theta = 0.$$

Therefore, statement A is wrong.

B. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \lim_{\theta\to 0} \sin\theta = 0.$$

However, as we see in D, the series $\sum a_n$ is divergent by Limit Comparison Test.

Therefore, statement B is wrong.

C. Set $b_n = \frac{1}{n}$.

Then $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 > 0.$$

However, $\sum b_n$ is divergent.

Therefore, statement C is wrong.

D. Set $b_n = \frac{1}{n}$.

Then $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 > 0$$

and $\sum b_n = \sum \frac{1}{n}$ is divergent.

Therefore, by Limit Comparison Test, $\sum a_n$ is divergent.

Therefore, statement D is correct.

E. Set $b_n = \frac{2}{n}$.

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{2}{n}}$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \frac{1}{2} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \frac{1}{2}.$$

This implies, for sufficiently large n, we have

$0 \leq a_n \leq b_n$, and NOT $0 \leq b_n \leq a_n$

Therefore, statement E is wrong.

$\boxed{\text{Answer D.}}$

7.

A. Set $b_n = \frac{n}{\sqrt{n^3+2}}$ and $c_n = \frac{n}{\sqrt{n^3}} = \frac{1}{n^{\frac{1}{2}}}$

Then $\lim_{n\to\infty} \frac{b_n}{c_n} = 1 > 0$ and $\sum c_n$ diverges.

Therefore, by Limit Comparison Test, $\sum b_n$ also diverges.

Therefore, statement A is wrong.

B. Set $b_n = \frac{n}{\sqrt{n^3+2}}$. and $c_n = \frac{n}{\sqrt{n^3}} = \frac{1}{n^{\frac{1}{2}}}$.

Then $\lim_{n\to\infty} \frac{b_n}{c_n} = 1 > 0$ and $\sum c_n$ diverges.

Therefore, by Limit Comparison Test, $\sum b_n$ also diverges.

However, the divergence of the series $\sum b_n = \sum |a_n|$ by itself does not imply anything. (There is NO such notion as "absolutely diverges".)

Therefore, statement B is wrong.

C. Set $b_n = \frac{n}{\sqrt{n^3+2}}$.

Then $b_n \geq b_{n+1}$ and $\lim_{n\to\infty} b_n = 0$.

On the other hand $\sum b_n$ diverges.

(See the reasoning in A or B above using Limit Comparison Test.)

Therefore, $\sum a_n$ converges by Alternating Series Test, while $\sum b_n = \sum |a_n|$ diverges.

Therefore, $\sum a_n$ conditionally converges.

Therefore, statement C is correct.

D. Set $b_n = \dfrac{n}{\sqrt{n^3+2}}$.

Then $b_n \geqq b_{n+1}$ and $\lim b_n = 0$.

Therefore, statement D is wrong.

E. Set $f(x) = \dfrac{x}{\sqrt{x^3+2}}$.

Then

$$0 \leqq \frac{x}{2\sqrt{x^3}} = \frac{x}{\sqrt{x^3+3x^3}} \leqq \frac{x}{\sqrt{x^3+2}}$$

and hence

$$\int_1^\infty \frac{x}{2\sqrt{x^3}}\,dx \leqq \int_1^\infty \frac{x}{\sqrt{x^3+2}}\,dx = \int_1^\infty f(x)\,dx.$$

On the other hand

$$\int_1^\infty \frac{x}{2\sqrt{x^3}}\,dx = \int_1^\infty \frac{1}{2x^{\frac{1}{2}}}\,dx = \infty.$$

Therefore, we have

$$\int_1^\infty f(x)\,dx = \infty.$$

Therefore, statement E is wrong.

$\boxed{\text{Answer C.}}$

8.

A. We compute

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{(n+1)^2\, 2^{n+1}}{(n+1)!}}{\frac{n^2\, 2^n}{n!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{2}{n+1} = 0$$

Therefore, by Ratio Test, $\sum a_n$ absolutely converges.

Therefore, statement A is correct.

B. As in A above, $\sum a_n$ absolutely converges. (i.e. $\sum |a_n|$ converges), while statement B claims that $\sum a_n$ conditionally converges.

Therefore, statement B is wrong.

C. As in A above, $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0$ and $\sum a_n$ absolutely converges.

Therefore, statement C is wrong.

D. As in A above, $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0$ and $\sum a_n$ absolutely converges.

Therefore, statement D is wrong.

E. Set $b_n = \frac{n^2 2^n}{n!}$.

Then in fact, $b_n \geq b_{n+1}$ for sufficiently large n, and $\lim_{n\to\infty} b_n = 0$.

However, $\sum b_n = \sum |a_n|$ converges.

Therefore, statement E is wrong.

$\boxed{\text{Answer A.}}$

9. The estimation theorem for the alternating series, $\sum (-1)^n b_n$ ($b_n \geq 0$) states, assuming (i) $b_n \geq b_{n+1}$ and (ii) $\lim_{n\to\infty} b_n = 0$, that

$$|S - S_k| \leq b_{k+1}.$$

Therefore, in order to guarantee

$$|S - S_k| \leq 10^{-7}$$

using the above theorem, we have to have

$$b_{k+1} = \frac{1}{10^{k+1}(k+1)!} \leq 10^{-7},$$

that is to say,

$$10^7 \leq 10^{k+1}(k+1)!$$

Now we carry out the following computation.

k	$k+1$	$10^{k+1}(k+1)!$
1	2	$10^2 \cdot 2! < 10^7$
2	3	$10^3 \cdot 3! < 10^7$
3	4	$10^4 \cdot 4! < 10^7$
4	5	$10^5 \cdot 5! > 10^7$
5	6	$10^6 \cdot 6! > 10^7$
$\vdots$	$\vdots$	$\vdots$

Therefore, we conclude that the least k which guarantees the condition is $k = 4$.

$\boxed{\text{Answer} \quad \text{B. } 4.}$

10. Set

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1} = \sum_{n=0}^{\infty} a_n$$

We compute

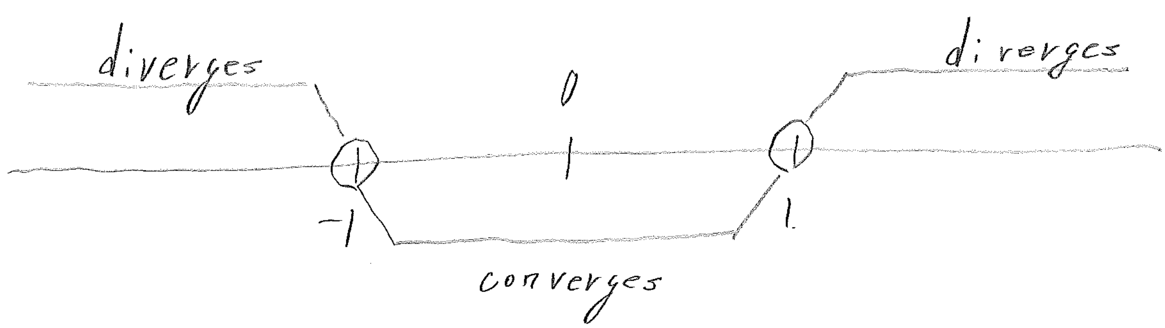
$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\dfrac{|x|^{n+1}}{n+2}}{\dfrac{|x|^n}{n+1}}$$

$$= \lim_{n\to\infty} \frac{n+1}{n+2} |x| = |x|.$$

By Ratio Test, we conclude

$|x| < 1 \Rightarrow \sum a_n$ (absolutely) converges.

$|x| > 1 \Rightarrow \sum a_n$ diverges.



When $x = 1$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$$ converges by Alternating Series Test.

When $x = -1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}$$ diverges.

Therefore, we conclude that the interval of convergence is $(-1, 1]$.

$\boxed{\text{Answer} \quad A. \ (-1, 1]}$

17. The interval of convergence I for $\sum_{n=0}^{\infty} c_n x^n$ is of the form

$$\overset{(}{\underset{[}{\text{or}}} \quad -R \ , \quad R \quad \overset{)}{\underset{]}{\text{or}}} \ ,$$

where R is the radius of convergence.

Since $-4 \in I$ and $-6 \notin I$, we conclude

$$4 \leq R \leq 6.$$

Therefore, we conclude

(i) $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} c_n x^n \quad (x = 1)$
convergent

(ii) $\sum_{n=0}^{\infty} c_n 8^n = \sum_{n=0}^{\infty} c_n x^n \quad (x = 8)$
divergent

(iii) $\sum_{n=0}^{\infty} c_n (-3)^n = \sum_{n=0}^{\infty} c_n x^n \quad (x = -3)$
convergent

(iv) $\sum_{n=0}^{\infty} (-1)^n c_n 9^n = \sum_{n=0}^{\infty} c_n x^n \quad (x = -9)$
divergent.

$\boxed{\text{Answer B. (i) and (iii) only.}}$

12.

$$f(x) = \frac{2}{3-x}$$

$$= \frac{2}{3} \cdot \frac{1}{1-\left(\frac{x}{3}\right)}$$

$$= \frac{2}{3}\left\{ 1 + \left(\frac{x}{3}\right) + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^3 + \cdots \right\}.$$

(Note:

$$\frac{1}{1-X} = 1 + X + X^2 + X^3 + \cdots \quad)$$

$$= \sum_{n=0}^{\infty} \frac{2}{3} \cdot \left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} a_n$$

We compute

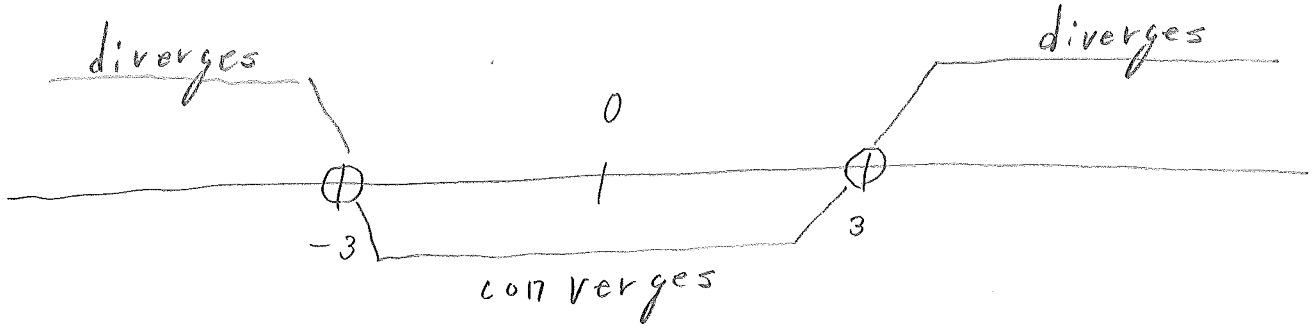
$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{2}{3}\,\frac{|x|^{n+1}}{3^{n+1}}}{\frac{2}{3}\,\frac{|x|^n}{3^n}}$$

$$= \lim_{n\to\infty} \frac{1}{3}|x| = \frac{1}{3}|x|.$$

By Ratio Test, we conclude

$\frac{1}{3}|x| < 1$ i.e. $|x| < 3 \Rightarrow \sum a_n$ (absolutely) converges

$\frac{1}{3}|x| > 1$ i.e. $|x| > 3 \Rightarrow \sum a_n$ diverges.



When $x = 3$,

$$\sum_{n=0}^{\infty} \frac{2}{3}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{2}{3} \quad \text{diverges.}$$

When $x = -3$

$$\sum_{n=0}^{\infty} \frac{2}{3}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{2}{3} \quad \text{diverges.}$$

Therefore, we conclude that the interval of convergence is $(-3, 3)$

$\boxed{\text{Answer D. } \frac{2}{3}\left(1 + \left(\frac{x}{3}\right) + \left(\frac{x}{3}\right)^2 + \cdots\right) \text{ and } (-3, 3)}$

13.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Therefore, we have

$$f'(x) = \frac{e^x - 1}{x}$$

$$= \frac{1}{1!} + \frac{x}{2!} + \frac{x^2}{3!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

Therefore, we compute

$$f(x) = \int f'(x)\, dx$$

$$= \int \left(\frac{1}{1!} + \frac{x}{2!} + \frac{x^2}{3!} + \cdots \right) dx$$

$$= x + \frac{x^2}{2(2!)} + \frac{x^3}{3(3!)} + \cdots + C.$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n(n!)} + C.$$

In order to determine C, we plug in $x = 0$ to have $f(0) = C = 5$.

Therefore, we conclude that the power series of $f(x)$ is given by

$$f(x) = 5 + x + \frac{x^2}{2(2!)} + \frac{x^3}{3(3!)} + \cdots$$

$$= 5 + \sum_{n=1}^{\infty} \frac{x^n}{n(n!)}$$

$$= 5 + \sum_{n=1}^{\infty} a_n .$$

Moreover, we compute

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{(n+1)\{(n+1)!\}}}{\frac{|x|^n}{n(n!)}}$$

$$= \lim_{n\to\infty} \frac{n}{n+1} |x| \cdot \frac{1}{n+1} = 0 .$$

Therefore, we conclude $R = \infty$.

$$\boxed{\text{Answer B.} \quad 5 + x + \frac{x^2}{2(2!)} + \frac{x^3}{3(3!)} + \frac{x^4}{4(4!)} + \cdots \quad \text{and } R = \infty}$$

14.

We compute

$$
\begin{aligned}
\frac{1}{x} &= \frac{1}{(x-3)+3} \\
&= \frac{1}{3-\{-(x-3)\}} \\
&= \frac{1}{3} \cdot \frac{1}{1-\left\{-\left(\frac{x-3}{3}\right)\right\}} \\
&= \frac{1}{3}\left[1+\left\{-\left(\frac{x-3}{3}\right)\right\}+\left\{-\left(\frac{x-3}{3}\right)\right\}^2 \right. \\
&\qquad \left. +\left\{-\left(\frac{x-3}{3}\right)\right\}^3+\left\{-\left(\frac{x-3}{3}\right)\right\}^4+\cdots\right] \\
&= \frac{1}{3}\left(1-\left(\frac{x-3}{3}\right)+\left(\frac{x-3}{3}\right)^2-\left(\frac{x-3}{3}\right)^3+\left(\frac{x-3}{3}\right)^4 \right. \\
&\qquad \left. -\cdots\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{3}\left\{-\left(\frac{x-3}{3}\right)\right\}^n \\
&= \sum_{n=0}^{\infty} a_n.
\end{aligned}
$$

Moreover, we compute

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{1}{3}\cdot\left(\frac{|x-3|}{3}\right)^{n+1}}{\frac{1}{3}\cdot\left(\frac{|x-3|}{3}\right)^{n}}$$

$$= \frac{|x-3|}{3}.$$

Since Ratio Test states

$$\begin{cases} \dfrac{|x-3|}{3} < 1 \text{ i.e. } |x-3| < 3 \\ \qquad\qquad \Rightarrow \sum a_n \text{ converges.} \\ \dfrac{|x-3|}{3} > 1 \text{ i.e. } |x-3| > 3 \\ \qquad\qquad \Rightarrow \sum a_n \text{ diverges,} \end{cases}$$

we conclude $R = 3$.

$$\boxed{\begin{array}{l} \text{Answer } C. \\ \frac{1}{3}\left(1 - \left(\frac{x-3}{3}\right) + \left(\frac{x-3}{3}\right)^2 - \left(\frac{x-3}{3}\right)^3 + \left(\frac{x-3}{3}\right)^4 - \cdots\right) \text{ and } R = 3 \end{array}}$$