Answer Keys for Version 02.

1. (i) $\lim_{n\to\infty} n e^{-n}$

$$= \lim_{n\to\infty} \frac{n}{e^{n}} = \lim_{x\to\infty} \frac{x}{e^{x}}$$

L'Hospital's Rule

$$= \lim_{x\to\infty} \frac{1}{e^{x}} = 0.$$

Therefore, the sequence $\{n e^{-n}\}_{n=1}^{\infty}$ is convergent.

(ii) $\lim_{n\to\infty} \sqrt{n^2+n} - n.$

$$= \lim_{n\to\infty} \frac{(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n)}{\sqrt{n^2+n} + n.}$$

$$= \lim_{n\to\infty} \frac{(n^2+n) - n^2}{\sqrt{n^2+n} + n.}$$

$$= \lim_{n\to\infty} \frac{n \,/\, n}{(\sqrt{n^2+n} + n)/n.}$$

$$= \lim_{n\to\infty} \frac{1.}{\sqrt{(n^2+n)/n^2} + 1.} = \frac{1}{2.}$$

Therefore, the sequence $\{\sqrt{n^2+n} - n\}_{n=1}^{\infty}$

is convergent.

(iii) $\lim_{n\to\infty} \{1+(-1)^n\}$. does not exist.

Therefore, the sequence $\{1+(-1)^n\}_{n=1}^{\infty}$

is divergent.

$\boxed{\text{Answer B. (i) and (ii) only}}$

2.

(i) Set $a_n = \ln\left(\frac{2n^2-1}{n^2+1}\right)$

Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{2n^2-1}{n^2+1}\right) = \ln 2 \neq 0.$$

Therefore, by Test for divergence, we conclude that the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \ln\left(\frac{2n^2-1}{n^2+1}\right) \text{ is divergent.}$$

(ii) Set $a_n = \arcsin\left(\frac{n}{2(n+1)}\right)$.

Then.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arcsin\left(\frac{n}{2(n+1)}\right)$$

$$= \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6} \neq 0.$$

Therefore, by Test for divergence, we conclude that the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \arcsin\left(\frac{n}{2(n+1)}\right) \text{ is divergent.}$$

(iii) Set $a_n = \frac{1+7^n}{5^n}$ and $b_n = \frac{7^n}{5^n}$.

Then we have

$$0 \leqq b_n = \frac{7^n}{5^n} \leqq \frac{1+7^n}{5^n} = a_n.$$

and $\sum b_n$ is divergent (since it is a geometric series with $r = \frac{7}{5} > 1$).

Therefore, we conclude, by Comparison Test, that $\sum a_n$ is divergent.

$\boxed{\text{Answer D. All.}}$

3. Set

$$a_n = \frac{1}{(n^7 + n^3)^p}$$

$$b_n = \frac{1}{n^{7p}}.$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{(n^7+n^3)^p}}{\frac{1}{n^{7p}}}$$

$$= \lim_{n\to\infty} \left(\frac{n^7}{n^7+n^3}\right)^p = 1.$$

Therefore, by Limit Comparison Test, the two series $\sum a_n$ and $\sum b_n$ share the same destiny.

On the other hand, the condition for the series $\sum b_n = \sum \frac{1}{n^{7p}}$ to converge is $1 < 7p$, i.e. $\frac{1}{7} < p$.

Therefore, we conclude that the condition on the value of p for which the series $\sum a_n$ converges is the same, i.e., $\frac{1}{7} < p$.

$$\boxed{\text{Answer E. } \frac{1}{7} < p}$$

4. A geometric series with the initial term a, and the ratio r $(-1 < r < 1)$ converges to $\frac{a}{1-r}$.

Therefore, we evaluate

$$\sum_{n=1}^{\infty} \frac{10 \cdot 8^{n-1} - 3^n}{10^n} = \sum_{n=1}^{\infty} \left(\frac{8}{10}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{3}{10}\right)^n$$

$$= \frac{1}{1-\frac{8}{10}} - \frac{3/10}{1-\frac{3}{10}} = 5 - \frac{3}{7} = \frac{32}{7}.$$

$$\boxed{\text{Answer D. } \frac{32}{7}}$$

5.

A. Set $b_n = \frac{1}{n^2}$.

Then

$$0 \leq a_n = \frac{\cos^2 n}{n^2+1} \leq \frac{1}{n^2+1} \leq b_n$$

and $\sum b_n = \sum \frac{1}{n^2}$ is convergent.

Therefore, by Comparison Test, $\sum a_n$ is convergent.

Therefore, statement A is correct.

B. Set $b_n = \frac{1}{n^2}$.

Then, actually $0 \leq a_n \leq b_n$ and $\sum b_n$ is convergent, and NOT divergent.

Therefore, statement B is wrong.

C. Set $b_n = \frac{1}{n^2}$.

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\cos^2 n}{n^2+1}}{\frac{1}{n^2}}$

$= \lim_{n\to\infty} \frac{n^2}{n^2+1} \cdot \cos^2 n$ does not exist.

Therefore, statement C is wrong.

D. Set $b_n = \dfrac{1}{n^2}$.

Then as in C, $\lim_{n\to\infty} \dfrac{a_n}{b_n}$ does not exist.

Therefore, statement D is wrong.

E. We have

$$0 \leqq a_n = \frac{\cos^2 n}{n^2+1} \leqq \frac{1}{n^2} = b_n$$

and $\lim_{n\to\infty} b_n = 0$.

Therefore, $\lim_{n\to\infty} a_n = 0$.

However, as in A, we conclude by Comparison Test that $\sum a_n$ is convergent.

Therefore, statement D is wrong.

$\boxed{\text{Answer A.}}$

6.

A. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \tan\frac{1}{n} = \lim_{\theta\to 0} \tan\theta = 0.$$

Therefore, statement A is wrong

B. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \tan\frac{1}{n} = \lim_{\theta\to 0} \tan\theta = 0.$$

However, as we see in D, the series $\sum a_n$ is divergent by Limit Comparison Test.

Therefore, statement B is wrong.

C. Set $b_n = \frac{1}{n}$.

Then $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}}.$$

$$= \lim_{\theta\to 0} \frac{\tan\theta}{\theta} = \lim_{\theta\to 0} \frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\theta} = 1. > 0$$

However, $\sum b_n$ is divergent.

Therefore, statement C is wrong.

D. Set $b_n = \frac{1}{n}$.

Then $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}}$$

$$= \lim_{\theta\to 0} \frac{\tan\theta}{\theta} = \lim_{\theta\to 0} \frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\theta} = 1 > 0$$

and $\sum b_n = \sum \frac{1}{n}$ is divergent.

Therefore, by Limit Comparison Test, $\sum a_n$ is divergent.

Therefore, statement D is correct.

E. Set $b_n = \frac{2}{n}$.

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\frac{2}{n}}$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}} = \frac{1}{2} \lim_{\theta\to 0} \frac{\tan\theta}{\theta}$$

$$= \frac{1}{2} \lim_{\theta\to 0} \frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\theta} = \frac{1}{2}.$$

This implies, for sufficiently large n, we have

$0 \leqq a_n \leqq b_n$, and NOT $0 \leqq b_n \leqq a_n$.

Therefore, statement E is wrong

$\boxed{\text{Answer. D.}}$

7.

A. Set $b_n = \dfrac{1}{n \ln n}$.

Then $\sum b_n$ diverges. (See C below.)

Therefore, statement A is wrong.

B. There is no such notion as "absolutely diverges".

Therefore, statement B is wrong.

C. Set $b_n = \dfrac{1}{n \ln n}$.

Then $b_n \geq b_{n+1}$ and $\lim_{n\to\infty} b_n = 0$.

On the other hand, $\sum b_n$ diverges.

Set $f(x) = \dfrac{1}{x \ln x}$. Then

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x \ln x}\,dx$$

$$= \lim_{t\to\infty} \int_2^t \frac{1}{x \ln x}\,dx$$

$$\left(\begin{array}{ll} u = \ln x & \\ du = \frac{1}{x}\,dx & \\ x & u \\ t & \ln t \\ 2 & \ln 2 \end{array} \right)$$

$$= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u}\,du = \lim_{t\to\infty} [\ln u]_{\ln 2}^{\ln t}$$

$$= \infty$$

Therefore, by Alternating Series Test,

$\sum a_n = \sum (-1)^n b_n$ converges, while $\sum |a_n| = \sum b_n$ diverges.

Therefore, $\sum a_n$ conditionally converges.

Therefore, statement C is correct.

D. Set $b_n = \frac{1}{n \ln n}$.

Then $b_n \geq b_{n+1}$ and $\lim_{n \to \infty} b_n = 0$.

Therefore, statement D is wrong.

E. Set $f(x) = \frac{1}{x \ln x}$

Then $\int_2^\infty f(x)\,dx = \infty$ as shown in C.

Therefore, statement E is wrong.

$\boxed{\text{Answer C.}}$

8.

A. We compute

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

Therefore, statement A, which claims

$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0$, is wrong.

B. As we computed above, $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = e$.

Therefore, statement B, which claims

$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0$, is wrong.

C. As we computed above,

$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = e$.

Therefore, by Ratio Test, $\sum a_n$ diverges.

Therefore, statement C is correct.

D. As we computed above, $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = e$.

Therefore, statement D, which claims

$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \infty$, is wrong.

E. Set $b_n = \frac{n^n}{n!}$.

Then $b_n < b_{n+1}$, and NOT $b_n \geq b_{n+1}$.

(In fact, $\frac{b_{n+1}}{b_n} = \left(\frac{n+1}{n}\right)^n > 1$.

See the computation in A above.)

Therefore, statement E is wrong.

$\boxed{\text{Answer C.}}$

9. The estimation theorem for the alternating series $\sum (-1)^n b_n$ $(b_n \geqq 0)$ states, assuming (i) $b_n \geqq b_{n+1}$ and (ii) $\lim_{n\to\infty} b_n = 0$, that

$$|S - S_k| \leqq b_{k+1}.$$

Therefore, in order to guarantee

$$|S - S_k| \leqq 10^{-6}$$

using the above theorem, we have to have

$$b_{k+1} = \frac{1}{10^{k+1}(k+1)!} \leqq 10^{-6},$$

that is to say,

$$10^6 \leqq 10^{k+1}(k+1)!$$

Now we carry out the following computation

k	$k+1$.	$10^{k+1}(k+1)!$
1	2	$10^2 \cdot 2! < 10^6$
2	3	$10^3 \cdot 3! < 10^6$
3	4	$10^4 \cdot 4! < 10^6$
4	5	$10^5 \cdot 5! > 10^6$
5	6	$10^6 \cdot 6! > 10^6$
$\vdots$		

Therefore, we conclude that the least k which guarentees the condition is $k = 4$.

$\boxed{\text{Answer B. 4.}}$

10. Set

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n^3} = \sum_{n=0}^{\infty} a_n.$$

We compute

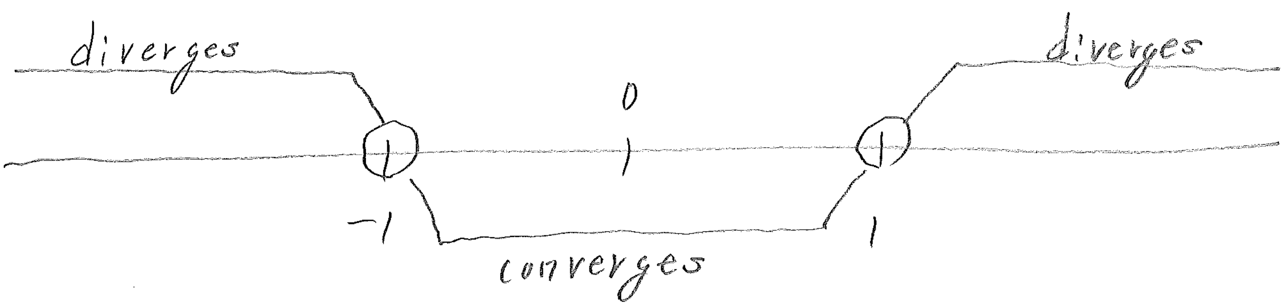
$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\dfrac{|x|^{n+1}}{(n+1)^3}}{\dfrac{|x|^n}{n^3}}$$

$$= \lim_{n\to\infty} \frac{n^3}{(n+1)^3} |x| = |x|.$$

By Ratio Test, we conclude

$|x| < 1 \Rightarrow \sum a_n$ (absolutely) converges

$|x| > 1 \Rightarrow \sum a_n$ diverges



When $x = 1$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n^3} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n^3}$$

(absolutely) converges.

When $x = -1$,

$$\sum_{n=0}^{\infty} \frac{(-1) x^n}{n^3} = \sum_{n=0}^{\infty} \frac{1}{n^3} \text{ converges.}$$

Therefore, we conclude that the interval of convergence is $[-1, 1]$

$\boxed{\text{Answer C. } [-1, 1]}$

11. The interval of convergence I for $\sum_{n=0}^{\infty} c_n x^n$ is of the form

$$\begin{matrix} (& & & &) \\ \text{or} & -R & , & R & \text{or} \\ [& & & &] \end{matrix}$$

where R is the radius of convergence.

Since $-6 \in I$ and $8 \notin I$, we conclude

$$6 \leq R \leq 8.$$

Therefore, we conclude

(i) $\displaystyle\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} c_n x^n \;(x = 1)$

convergent

(ii) $\displaystyle\sum_{n=0}^{\infty} c_n 5^n = \sum_{n=0}^{n} c_n x^n \;(x = 5)$

convergent

(iii) $\displaystyle\sum_{n=0}^{\infty} c_n (-3)^n = \sum_{n=0}^{\infty} c_n x^n \;(x = -3)$

convergent

(iv) $\displaystyle\sum_{n=0}^{\infty} (-1)^n c_n 9^n = \sum_{n=0}^{\infty} c_n x^n \;(x = -9)$

divergent.

$\boxed{\text{Answer B. (i), (ii) and (iii) only.}}$

12.

$$f(x) = \frac{x}{9+x^2}$$

$$= \frac{x}{9} \cdot \frac{1}{1-\left(-\frac{x^2}{9}\right)}$$

$$= \frac{x}{9}\left\{ 1 + \left(-\frac{x^2}{9}\right) + \left(-\frac{x^2}{9}\right)^2 + \left(-\frac{x^2}{9}\right)^2 + \cdots \right\}$$

(Note :

$$\frac{1}{1-X} = 1 + X + X^2 + X^3 + \cdots$$
)

$$= \sum_{n=0}^{\infty} \frac{x}{9}\left(-\frac{x^2}{9}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{9^{n+1}} x^{2n+1}$$

We compute

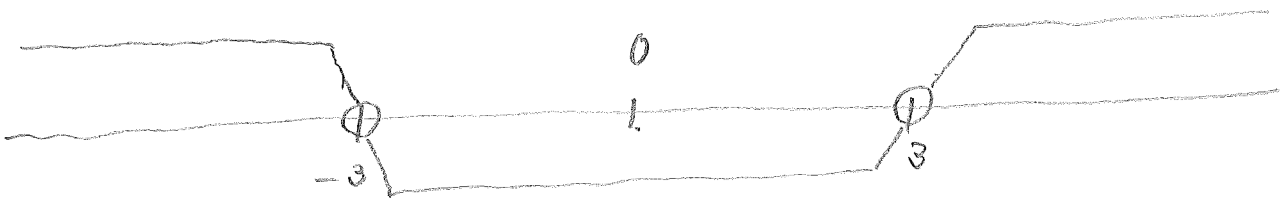
$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{1}{9^{(n+1)+1}} |x|^{2(n+1)+1}}{\frac{1}{9^{n+1}} |x|^{2n+1}}$$

$$= \lim_{n\to\infty} \frac{|x|^2}{9} = \frac{|x|^2}{9}.$$

By Ratio Test, we conclude

$\frac{|x|^2}{9} < 1$ i.e. $|x| < 3 \Rightarrow \sum a_n$ (absolutely) converges

$\frac{|x|^2}{9} > 1$ i.e. $|x| > 3 \Rightarrow \sum a_n$ diverges



When $x = 3$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{9^{n+1}} x^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3} \text{ diverges}$$

When $x = -3$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{9^{n+1}} x^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \left(-\frac{1}{3}\right) \text{ diverges.}$$

Therefore, we conclude that the interval of convergence is $(-3, 3)$

$$\boxed{\text{Answer } B. \quad \frac{x}{9}\left(1 - \frac{x^2}{9} + \left(\frac{x^2}{9}\right)^2 - \left(\frac{x^2}{9}\right)^3 + \cdots\right) \text{ and } (-3, 3)}$$

13. $$\frac{1}{1-x^8} = 1 + x^8 + (x^8)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} (x^8)^n = \sum_{n=0}^{\infty} x^{8n}$$

Therefore, we have

$$f'(x) = \frac{x}{1-x^8}$$

$$= x + x \cdot x^8 + x(x^8)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} x^{8n+1}$$

Therefore, we compute

$$f(x) = \int f'(x)\,dx$$

$$= \int \left(x + x^9 + x^{17} + x^{25} + \cdots\right) dx$$

$$= \frac{x^2}{2} + \frac{x^{10}}{10} + \frac{x^{18}}{18} + \frac{x^{26}}{26} + \cdots + C$$

$$= C + \frac{x^2}{2} + \frac{x^{10}}{10} + \frac{x^{18}}{18} + \frac{x^{26}}{26} + \cdots$$

$$= C + \sum_{n=0}^{\infty} \frac{1}{8n+2} x^{8n+2}$$

In order to determine C, we plug in $x=0$ to have

$$f(0) = C = 7.$$

Therefore, we conclude that the power series of $f(x)$ is given by

$$f(x) = 7 + \frac{x^2}{2} + \frac{x^{10}}{10} + \frac{x^{18}}{18} + \frac{x^{26}}{26} + \cdots$$

$$= 7 + \sum_{n=0}^{\infty} \frac{1}{8n+2} x^{8n+2}$$

Moreover, we compute

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{1}{8(n+1)+2}|x|^{8(n+1)+2}}{\frac{1}{8n+2}|x|^{8n+2}}$$

$$= \lim_{n\to\infty} \frac{8n+2}{8n+10}|x|^8 = |x|^8$$

By Ratio Test, we conclude

$|x|^8 < 1$ i.e. $|x| < 1$

$\Rightarrow \sum a_n$ (absolutely) converges.

$|x|^8 > 1$ i.e. $|x| > 1$

$\Rightarrow \sum a_n$ diverges.

Therefore, we conclude $R = 1$

$$\boxed{\text{Answer B.} \quad 7 + \frac{x^2}{2} + \frac{x^{10}}{10} + \frac{x^{18}}{18} + \frac{x^{26}}{26} + \cdots \quad \text{and } R = 1.}$$

14.

We compute

$$
\begin{aligned}
\frac{1}{x} &= \frac{1}{(x-3)+3} \\
&= \frac{1}{3-\{-(x-3)\}} \\
&= \frac{1}{3} \cdot \frac{1}{1-\{-(\frac{x-3}{3})\}} \\
&= \frac{1}{3}\left[1+\left\{-\left(\frac{x-3}{3}\right)\right\}+\left\{-\left(\frac{x-3}{3}\right)\right\}^2 \right. \\
&\qquad \left. +\left\{-\left(\frac{x-3}{3}\right)\right\}^3+\left\{-\left(\frac{x-3}{3}\right)\right\}^4+\cdots\right] \\
&= \frac{1}{3}\left(1-\left(\frac{x-3}{3}\right)+\left(\frac{x-3}{3}\right)^2-\left(\frac{x-3}{3}\right)^3+\left(\frac{x-3}{3}\right)^4 \right. \\
&\qquad \left. -\cdots\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{3}\left\{-\left(\frac{x-3}{3}\right)\right\}^n \\
&= \sum_{n=0}^{\infty} a_n.
\end{aligned}
$$

Moreover, we compute

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{1}{3}\cdot\left(\frac{|x-3|}{3}\right)^{n+1}}{\frac{1}{3}\cdot\left(\frac{|x-3|}{3}\right)^{n}}$$

$$= \frac{|x-3|}{3}.$$

Since

$$\frac{|x-3|}{3} < 1 \quad \text{i.e.,} \quad |x-3| < 3$$
$$\Rightarrow \sum a_n \text{ converges}$$

$$\frac{|x-3|}{3} > 1 \quad \text{i.e.,} \quad |x-3| > 3$$
$$\Rightarrow \sum a_n \text{ diverges,}$$

we conclude $R = 3$.

$$\boxed{\begin{array}{l}\text{Answer E.}\\ \frac{1}{3}\left(1-\left(\frac{x-3}{3}\right)+\left(\frac{x-3}{3}\right)^2-\left(\frac{x-3}{3}\right)^3+\left(\frac{x-3}{3}\right)^4 - \cdots\right) \text{ and } R = 3.\end{array}}$$