Answer Keys

1. The equation of the sphere with center (a, b, c) and radius r is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

Since the sphere of the problem has its center $(1, 2, 3)$, the equation is of the form

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = r^2.$$

Since it passes the origin $(0, 0, 0)$, we have

$$\underbrace{(0-1)^2 + (0-2)^2 + (0-3)^2}_{\substack{\parallel \\ 14.}} = r^2$$

Therefore, the answer is

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 14.$$

01. B.	02. B.

2. The condition for the vectors

$$\vec{u} = \langle -6, b, 2 \rangle$$
$$\vec{v} = \langle b, b^2, b \rangle$$

to be orthogonal is given by

$$\begin{aligned}\vec{u} \cdot \vec{v} &= -6b + b^3 + 2b \\ &= b(b^2 - 4) \\ &= b(b-2)(b+2) = 0.\end{aligned}$$

Therefore, the values of b are.

$$b = 0, 2, -2.$$

$$\boxed{01.\ C \qquad\qquad 02.\ D.}$$

3. The formula for the vector projection $\text{proj}_{\vec{a}} \vec{b}$ is

$$\text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a}$$

$$= \frac{\langle 3, -4 \rangle \cdot \langle 5, 0 \rangle}{\langle 3, -4 \rangle \cdot \langle 3, -4 \rangle} \langle 3, -4 \rangle$$

$$= \frac{15}{25} \langle 3, -4 \rangle$$

$$= \frac{3}{5} \langle 3, -4 \rangle$$

01. C	02. B.

4. The formula for work is

$$W = \vec{F} \cdot \vec{D}$$

$$= |\vec{F}|\,|\vec{D}| \cos\theta$$

$$= 30 \times 100 \times \cos 30^\circ$$

$$= 30 \times 100 \times \frac{\sqrt{3}}{2} = 1500\sqrt{3}\ \mathrm{J}.$$

01.	E	02.	D

5. Consider the vectors

$$\overrightarrow{PQ} = \langle -3, 2, -1 \rangle$$
$$\overrightarrow{PR} = \langle 1, -1, 1 \rangle.$$

Then the cross product is orthogonal to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, and hence to the plane through the points P, Q, R

$$\overrightarrow{PQ} \times \overrightarrow{PR}$$

$$= \left\langle \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix}, - \begin{vmatrix} -3 & -1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix} \right\rangle$$

$$= \langle 1, 2, 1 \rangle$$

$$= i + 2j + k.$$

$$2 \overrightarrow{PQ} \times \overrightarrow{PR} = 2i + 4j + 2k.$$

is also orthogonal to the plane.

$$\boxed{01.\ D \qquad 02.\ B.}$$

6. The condition for the vectors

$$\vec{a} = \langle 1, 0, x \rangle$$
$$\vec{b} = \langle 2, x, 1 \rangle$$
$$\vec{c} = \langle 6, 1, 5 \rangle$$

to be coplanar is given by

$$\vec{a} \cdot (\vec{b} \times c) = \begin{vmatrix} 1 & 0 & x \\ 2 & x & 1 \\ 6 & 1 & 5 \end{vmatrix}$$

$$= 1 \times \begin{vmatrix} x & 1 \\ 1 & 5 \end{vmatrix} - 0 \times \begin{vmatrix} 2 & 1 \\ 6 & 5 \end{vmatrix} + x \times \begin{vmatrix} 2 & x \\ 6 & 1 \end{vmatrix}$$

$$= (5x - 1) + x(2 - 6x)$$

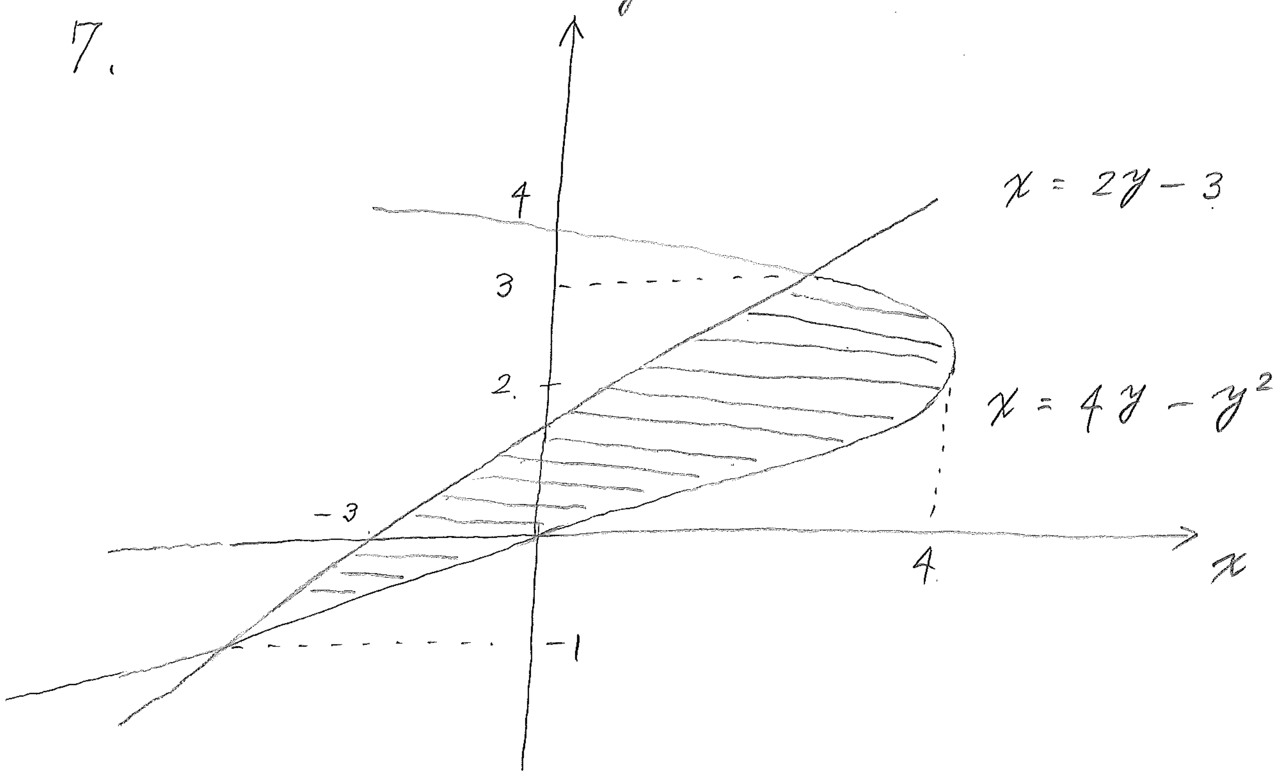
$$= -6x^2 + 7x - 1$$

$$= -(6x - 1)(x - 1) = 0.$$

Therefore, the values of x are $1/6$ and 1.

$$\boxed{01.\ C \qquad\qquad 02.\ E.}$$

7.



In order to compute the y-coordinate(s) of the intersection point(s) of the two curves

$$x = 4y - y^2 = -(y-2)^2 + 4.$$

$$x = 2y - 3,$$

we solve the equation

$$4y - y^2 = 2y - 3$$

i.e.

$$y^2 - 2y - 3 = (y+1)(y-3).$$

Therefore, we have $y = -1, 3.$

Therefore, the area of the region enclosed by the curves is given by

$$A = \int_{-1}^{3} \{(4y - y^2) - (2y - 3)\}\, dy$$

$$= \int_{-1}^{3} \{-y^2 + 2y + 3\}\, dy$$

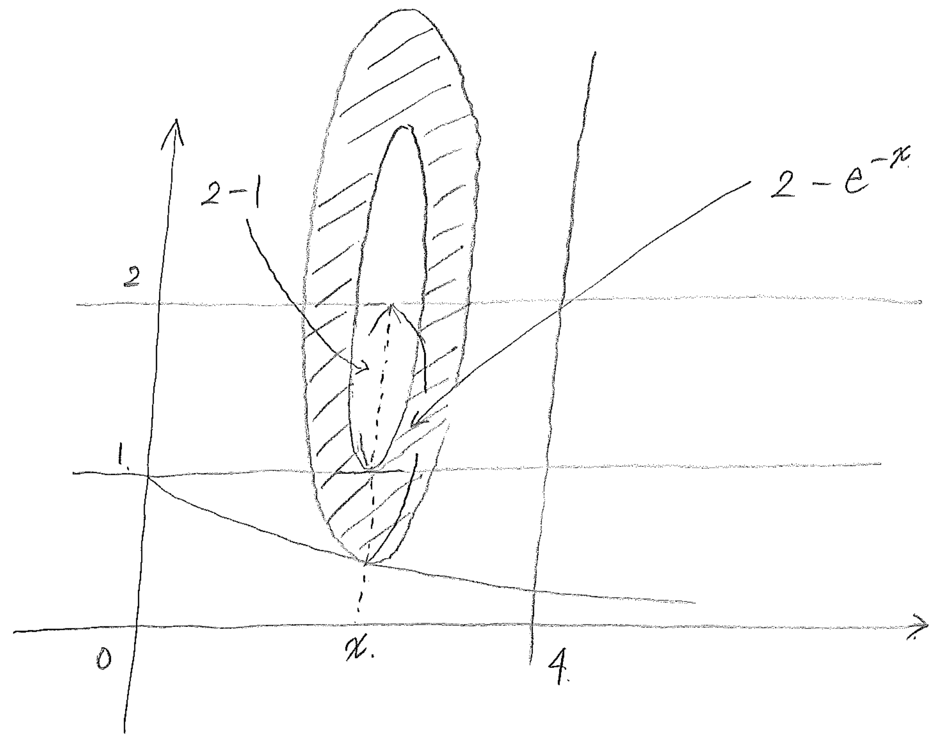
$$= \left[-\frac{y^3}{3} + y^2 + 3y\right]_{-1}^{3}$$

$$= \left[\left(-\frac{27}{3} + 9 + 9\right) - \left(-\frac{-1}{3} + 1 - 3\right)\right]$$

$$= \frac{32}{3}.$$

01. B	02. B.

8.



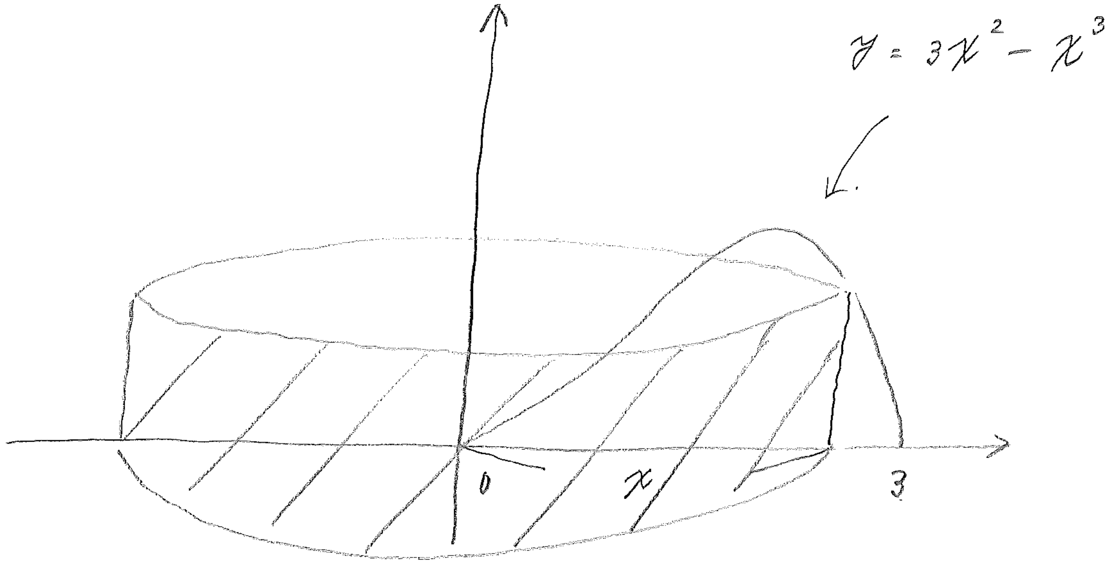
We compute the volume using the washer method. The cross-section at level x is in the shape of a washer, whose outer circle has radius $2-e^{-x}$ and whose inner circle has radius $2-1$. Therefore, the formula for the volume is given by

$$V = \int_0^4 \{\pi(2-e^{-x})^2 - \pi(2-1)^2\}\,dx$$

$$= \pi \int_0^4 (e^{-2x} - 4e^{-x} + 3)\,dx$$

01.	D	02	D.

9.



We compute the volume by the method of cylindrical shells. The cylindrical shell at level x has radius x and height $3x^2 - x^3$. Therefore, the volume is

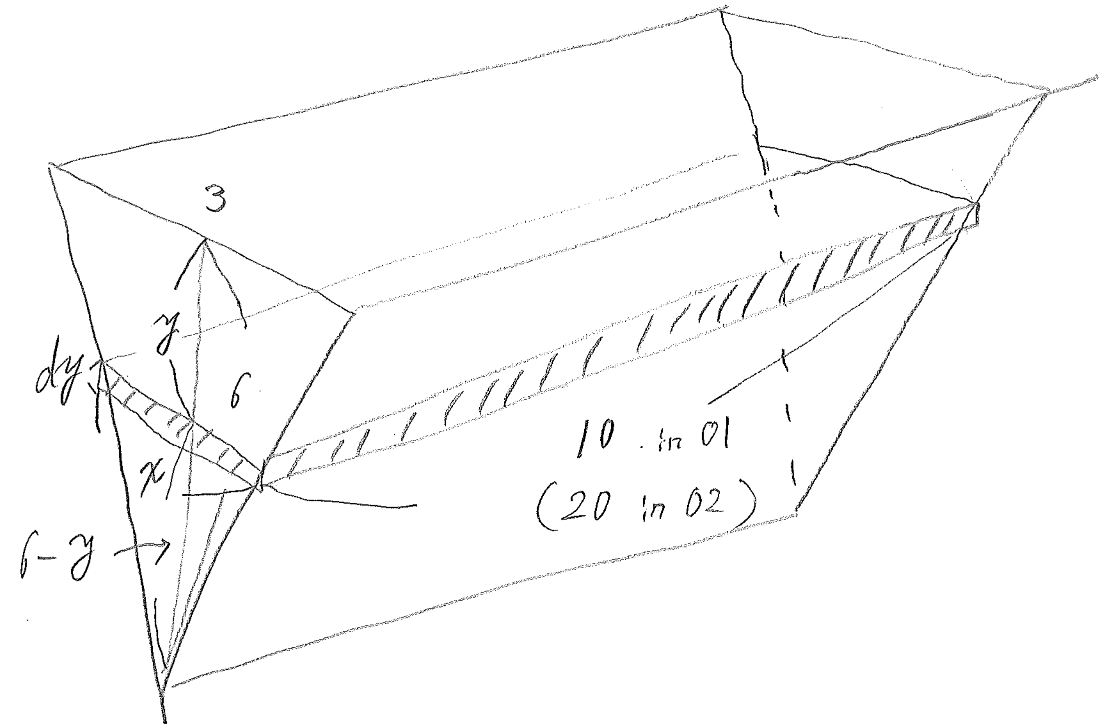
$$V = \int_0^3 2\pi x \,(3x^2 - x^3)\, dx$$

$$= 2\pi \int_0^3 (3x^3 - x^4)\, dx$$

$$= 2\pi \left[\frac{3}{4}x^4 - \frac{x^5}{5} \right]_0^3 = \frac{243}{10}\pi.$$

01.	F.	02.	F.

10.



The weight of the "slice" of water at level y from the top is

(☆) $$w = \underset{\substack{\uparrow \\ \text{gravitational} \\ \text{constant}}}{g} \cdot \underset{\substack{\uparrow \\ \text{density}}}{\delta} \cdot \underbrace{x \cdot 10 \cdot dy}_{\text{volume of the slice.}}$$

Now by looking at the two similar triangles of bottom lengths 3 and x, and of heights 6 and $6-y$, we have

$$3 : x = 6 : (6-y)$$

i.e. $$\frac{3}{x} = \frac{6}{6-y}$$

Therefore, we have

$$x = 3 \cdot \frac{6-y}{6} = \frac{1}{2}(6-y)$$

Plugging this and

$$g = 10, \quad s = 1000$$

into (☆), we obtain

$$w = 10 \times 1000 \times \frac{1}{2}(6-y) \times 10 \times dy$$
$$= 5 \times 10^{4}(6-y)\,dy$$

Therefore, the work done necessary to pump the water is

$$W = \int_0^6 w \cdot y = \int_0^6 5 \times 10^4 \times (6-y)y\,dy$$
$$= 5 \times 10^4 \times \int_0^6 (6y - y^2)\,dy$$
$$= 5 \times 10^4 \times \left[6 \cdot \frac{y^2}{2} - \frac{y^3}{3}\right]_0^6$$
$$= 5 \times 10^4 \times 36$$
$$= 18 \times 10^5 \text{ J}$$

$\boxed{01. \quad A}$

In 02, $L = 20$ instead of $L = 10$ in 01. Following the same calculation, we have

$$W = 36 \times 10^5 \text{ J}.$$

$\boxed{02 \quad B}$

11. We use integration by parts in order to compute

$$\int \sin^{-1}(5x)\,dx.$$

Set

$$\left(\begin{array}{ll} u = \sin^{-1}(5x) & dv = dx \\ du = \dfrac{5}{\sqrt{1-(5x)^2}} & v = x. \end{array} \right)$$

Then

$$\int \sin^{-1}(5x)\,dx.$$

$$= \int u\,dv = uv - \int v\,du.$$

$$(\star) \quad = \sin^{-1}(5x)\cdot x - \int \frac{5x}{\sqrt{1-(5x)^2}}\,dx.$$

On the other hand, we compute

$$\int \frac{5x}{\sqrt{1-(5x)^2}}\,dx.$$

using integration by substitution.

Set

$$t = 1 - (5x)^2 = 1 - 25x^2$$

$$dt = -50x \cdot dx$$

Therefore, we have

$$(\star\star) \int \frac{5x}{\sqrt{1-(5x)^2}}\,dx = \int \frac{-\frac{1}{10}\,dt}{\sqrt{t}}$$

$$= -\frac{1}{10}\int t^{-\frac{1}{2}}\,dt = -\frac{1}{10}\cdot 2t^{\frac{1}{2}} + c'$$

$$= -\frac{1}{5}\sqrt{1-(5x)^2} + c'$$

Plugging $(\star\star)$ back into $(\star)$ and setting $C = -C'$, we obtain

$$\int \sin^{-1}(5x)\,dx$$

$$= x\sin^{-1}(5x) + \frac{1}{5}\sqrt{1-(5x)^2} + C.$$

$$\boxed{01.\ E \qquad\qquad 02.\ A.}$$

12. We evaluate the integral using integration by parts

$$\int_1^{\sqrt{3}} \tan^{-1}\left(\frac{1}{x}\right) dx$$

$$\left(\begin{array}{ll} u = \tan^{-1}\left(\frac{1}{x}\right) & dv = dx \\ du = \dfrac{-\frac{1}{x^2}}{1+\left(\frac{1}{x}\right)^2} dx & v = x \\ \quad = -\dfrac{1}{1+x^2} dx. & \end{array}\right)$$

$$= \int_1^{\sqrt{3}} u\, dv$$

$$= [uv]_1^{\sqrt{3}} - \int_1^{\sqrt{3}} v\, du.$$

$$(\star) \quad = \left[\tan^{-1}\left(\frac{1}{x}\right) \cdot x\right]_1^{\sqrt{3}} - \int_1^{\sqrt{3}} \left(-\frac{x}{1+x^2}\right) dx.$$

On the other hand, we have

$$(\star\star) \quad \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}, \quad \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}.$$

Moreover, using integration by substitution, we compute

$$(☆☆☆) \int_1^{\sqrt{3}} \frac{x}{1+x^2}\,dx = \int_2^4 \frac{\frac{1}{2}\,dt}{t}$$

$$\left(\begin{array}{l} t = 1+x^2 \\ dt = 2x\,dx \end{array} \qquad \begin{array}{cc} x & t \\ \sqrt{3} & 4 \\ 1 & 2 \end{array} \right)$$

$$= \frac{1}{2}\int_2^4 \frac{1}{t}\,dt = \frac{1}{2}\left[\ln t\right]_2^4 = \frac{1}{2}(\ln 4 - \ln 2)$$

$$= \frac{1}{2}\ln\frac{4}{2} = \frac{1}{2}\ln 2.$$

Plugging (☆☆) and (☆☆☆) back into (☆), we have

$$\int_1^{\sqrt{3}} \tan^{-1}\left(\frac{1}{x}\right) dx$$

$$= \left(\frac{\pi}{6}\cdot\sqrt{3} - \frac{\pi}{4}\cdot 1\right) + \frac{1}{2}\ln 2.$$

$$\boxed{01.\ \ A \qquad\qquad 02.\ \ E.}$$