# CHAPTER 3
# Applications of the Derivative

# C H A P T E R 3
# Applications of the Derivative

## Section 3.1 Increasing and Decreasing Functions

### Solutions to Even-Numbered Exercises

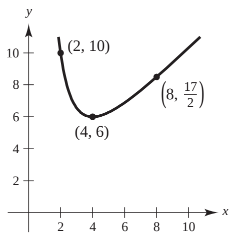
**2.** $f(x) = x + \dfrac{32}{x^2} = x + 32x^{-2}$

$f'(x) = 1 - 64x^{-3} = \dfrac{x^3 - 64}{x^3}$

At $(2, 10)$, $f$ is decreasing since $f'(2) = -7$.

At $(4, 6)$, $f$ has a critical number since $f'(4) = 0$.

At $\left(8, \frac{17}{2}\right)$, $f$ is increasing since $f'(8) = \frac{7}{8}$.




**4.** $f(x) = -3x\sqrt{x + 1} = -3x(x + 1)^{1/2}$

$$f'(x) = -3x\left[\frac{1}{2}(x + 1)^{-1/2}(1)\right] + (x + 1)^{1/2}(-3)$$

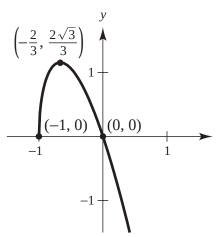
$$= -\frac{3}{2}(x + 1)^{-1/2}[x + 2(x + 1)]$$

$$= -\frac{3}{2}(x + 1)^{-1/2}(3x + 2) = -\frac{3(3x + 2)}{2\sqrt{x + 1}}$$

At $(-1, 0)$, $f$ has a critical number since $f'(-1)$ is undefined.

At $\left(-2/3, 2\sqrt{3}/3\right)$, $f$ has a critical number since $f'(-2/3) = 0$.

At $(0, 0)$, $f$ is decreasing since $f'(0) = -3$.




**6.** $f' = \dfrac{3}{4}x^2 - 3 = \dfrac{3}{4}(x^2 - 4)$

$f$ has critical numbers at $x = \pm 2$. Moreover, $f$ is increasing on $(-\infty, -2)$, $(2, \infty)$ and decreasing on $(-2, 2)$.

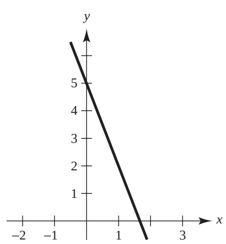
**8.** $f' = \dfrac{(x + 1)(2x) - (x^2)(1)}{(x + 1)^2} = \dfrac{x^2 + 2x}{(x + 1)^2} = \dfrac{x(x + 2)}{(x + 1)^2}$

$f$ has critical numbers at $x = 0, -2$ and has a discontinuity at $x = -1$. Moreover, $f$ is increasing on $(-\infty, -2)$, $(0, \infty)$ and decreasing on $(-2, -1)$, $(-1, 0)$.

**10.** $f(x) = 5 - 3x$

$f'(x) = -3$

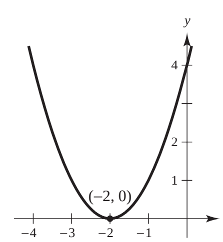
Since the derivative is negative for all $x$, the function is decreasing for all $x$. Thus, there are no critical numbers. $f$ is decreasing on $(-\infty, \infty)$.




**12.** $g(x) = (x + 2)^2$

$g'(x) = 2(x + 2) = 0$

Critical number: $x = -2$



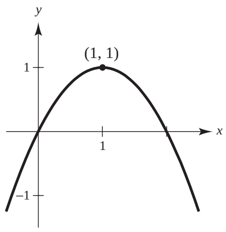

| Interval | $-\infty < x < -2$ | $-2 < x < \infty$ |
|---|---|---|
| Sign of $g'$ | $g' < 0$ | $g' > 0$ |
| Conclusion | Decreasing | Increasing |

**14.** $y = -(x^2 - 2x) = 2x - x^2$

$y' = 2 - 2x = 0$

Critical number: $x = 1$

| Interval | $-\infty < x < 1$ | $1 < x < \infty$ |
|---|---|---|
| Sign of $y'$ | $y' > 0$ | $y' < 0$ |
| Conclusion | Increasing | Decreasing |



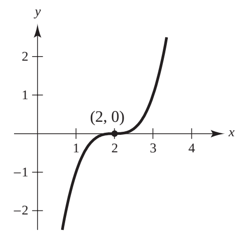

**16.** $y = (x - 2)^3$

$y' = 3(x - 2)^2 = 0$

Critical number: $x = 2$

| Interval | $-\infty < x < 2$ | $2 < x < \infty$ |
|---|---|---|
| Sign of $y'$ | $y' > 0$ | $y' > 0$ |
| Conclusion | Increasing | Increasing |

$y$ is increasing on $(-\infty, \infty)$.



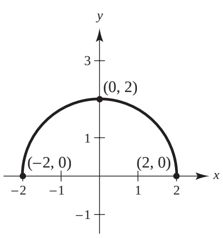

**18.** $f(x) = \sqrt{4 - x^2}$

$$f'(x) = \frac{1}{2}(4 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{4 - x^2}}$$

Domain: $-2 \le x \le 2$

Critical numbers: $x = 0, x = \pm 2$

| Interval | $-2 < x < 0$ | $0 < x < 2$ |
|---|---|---|
| Sign of $f'$ | $f' > 0$ | $f' < 0$ |
| Conclusion | Increasing | Decreasing |

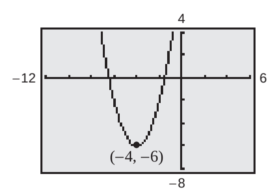



**20.** $f(x) = x^2 + 8x + 10$

$f'(x) = 2x + 8 = 0$

Critical number: $x = -4$

| Interval | $-\infty < x < -4$ | $-4 < x < \infty$ |
|---|---|---|
| Sign of $f'$ | $f' < 0$ | $f' > 0$ |
| Conclusion | Decreasing | Increasing |

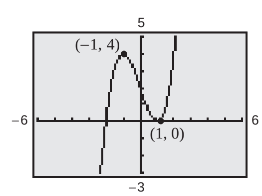



**22.** $f(x) = x^3 - 3x + 2 = (x - 1)^2(x + 2)$

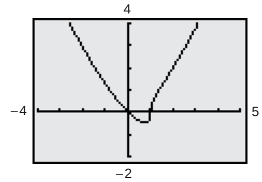
$f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$

Critical numbers: $x = \pm 1$

| Interval | $-\infty < x < -1$ | $-1 < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|
| Sign of $f'$ | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion | Increasing | Decreasing | Increasing |



**24.** $h(x) = x(x - 1)^{1/3}$




$$h'(x) = x\frac{1}{3}(x - 1)^{-2/3} + (x - 1)^{1/3}$$

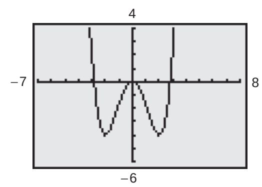
$$= \frac{1}{3}(x - 1)^{-2/3}(x + 3(x - 1))$$

$$= \frac{4x - 3}{3(x - 1)^{2/3}}$$

Critical numbers: $x = \frac{3}{4}, 1$

| Interval | $-\infty < x < \frac{3}{4}$ | $\frac{3}{4} < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|
| Sign of $h'$ | $h' < 0$ | $h' > 0$ | $h' > 0$ |
| Conclusion | Decreasing | Increasing | Increasing |

**26.** $f(x) = \frac{1}{4}x^4 - 2x^2$




$f'(x) = x^3 - 4x = x(x - 2)(x + 2)$

Critical numbers: $x = 0, 2, -2$

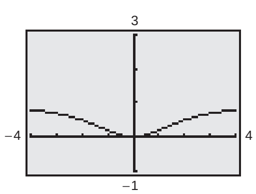
| Interval | $(-\infty, -2)$ | $(-2, 0)$ | $(0, 2)$ | $(2, \infty)$ |
|---|---|---|---|---|
| Sign of $f'$ | $f' < 0$ | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion | Decreasing | Increasing | Decreasing | Increasing |

**28.** $f(x) = y = \frac{x^2}{x^2 + 4}$

$$y' = \frac{(x^2 + 4)(2x) - (x^2)(2x)}{(x^2 + 4)^2} = \frac{8x}{(x^2 + 4)^2}$$

Critical number: $x = 0$

| Interval | $-\infty < x < 0$ | $0 < x < \infty$ |
|---|---|---|
| Sign of $f'$ | $f' < 0$ | $f' > 0$ |
| Conclusion | Decreasing | Increasing |

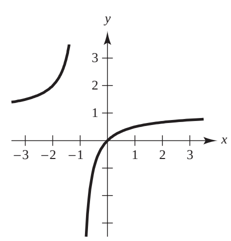



**30.** $f(x) = \frac{x}{x + 1}$

$$f'(x) = \frac{(x + 1)(1) - (x)(1)}{(x + 1)^2} = \frac{1}{(x + 1)^2}$$

No critical numbers
Discontinuity: $x = -1$

| Interval | $-\infty < x < -1$ | $-1 < x < \infty$ |
|---|---|---|
| Sign of $f'$ | $f' > 0$ | $f' > 0$ |
| Conclusion | Increasing | Increasing |

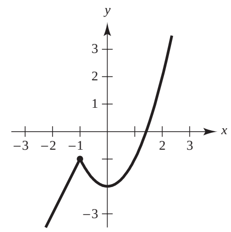


**32.** $y = f(x) = \begin{cases} 2x + 1, & x \le -1 \\ x^2 - 2, & x > -1 \end{cases}$

$f'(x) = \begin{cases} 2, & x < -1 \\ 2x, & x > -1 \end{cases}$

$f'(-1)$ is undefined.

Critical numbers: $x = -1, 0$



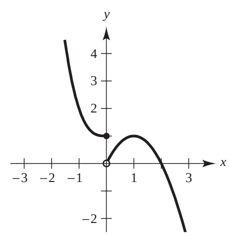

| Interval | $-\infty < x < -1$ | $-1 < x < 0$ | $0 < x < \infty$ |
|---|---|---|---|
| Sign of $f'$ | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion | Increasing | Decreasing | Increasing |

**34.** $y = f(x) = \begin{cases} -x^3 + 1, & x \le 0 \\ -x^2 + 2x, & x > 0 \end{cases}$

$f'(x) = \begin{cases} -3x^2, & x < 0 \\ -2x + 2, & x > 0 \end{cases}$

$f'(0)$ is undefined.

Critical numbers: $x = 0, 1$




| Interval | $-\infty < x < 0$ | $0 < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|
| Sign of $f'$ | $f' < 0$ | $f' > 0$ | $f' < 0$ |
| Conclusion | Decreasing | Increasing | Decreasing |

**36.** As the temperature increases, the average velocity (the curve's peak) and the spread of velocities increases. For each given temperature, the number of $N_2$ molecules increases for the first half of the interval of velocities and decreases for the second half of the interval.

**38.** $s(t) = -16t^2 + 64t, \quad 0 \le t \le 4$

$s'(t) = -32t + 64 = 0$

The critical number is $t = 2$. Therefore, the ball is moving up on the interval $(0, 2)$ and moving down on $(2, 4)$.

**40.** $P = 2.36x - \dfrac{x^2}{25{,}000} - 3500 \quad 0 \le x \le 50{,}000$

(a) $P' = 2.36 - \dfrac{1}{12{,}500}x = 0$

$x = (2.36)(12{,}500) = 29{,}500$

(b) You should charge the price that yields sales of $x = 29{,}500$ bags of popcorn.

## Section 3.2 Extrema and the First-Derivative Test

**2.** $f(x) = x^2 + 8x + 10$

$f'(x) = 2x + 8 = 2(x + 4)$

Critical number: $x = -4$

| Interval | $(-\infty, -4)$ | $(-4, \infty)$ |
|---|---|---|
| Sign of $f'$ | $-$ | $+$ |
| $f$ | Decreasing | Increasing |

Relative minimum: $(-4, -6)$

**4.** $f(x) = -4x^2 + 4x + 1$

$f'(x) = -8x + 4 = -4(2x - 1)$

Critical number: $x = \frac{1}{2}$

| Interval | $\left(-\infty, \frac{1}{2}\right)$ | $\left(\frac{1}{2}, \infty\right)$ |
|---|---|---|
| Sign of $f'$ | $f' > 0$ | $f' < 0$ |
| Conclusion | Increasing | Decreasing |

Relative maximum: $\left(\frac{1}{2}, 2\right)$

**6.** $g(x) = \frac{1}{5}x^5 - x = \frac{1}{5}(x^5 - 5x)$

$g'(x) = \frac{1}{5}(5x^4 - 5) = x^4 - 1$

Critical numbers: $x = \pm 1$

| Interval | $(-\infty, -1)$ | $(-1, 1)$ | $(1, \infty)$ |
|---|---|---|---|
| Sign of $g'$ | + | − | + |
| $g$ | Increasing | Decreasing | Increasing |

Relative maximum: $\left(-1, \frac{4}{5}\right)$

Relative minimum: $\left(1, -\frac{4}{5}\right)$

**8.** $h(x) = 2(x - 3)^3$

$h'(x) = 6(x - 3)^2$

Critical number: $x = 3$

| Interval | $(-\infty, 3)$ | $(3, \infty)$ |
|---|---|---|
| Sign of $h'$ | + | + |
| $h$ | Increasing | Increasing |

No relative extrema

**10.** $f(x) = x^4 - 32x + 4$

$f'(x) = 4x^3 - 32 = 4(x^3 - 8)$

Critical number: $x = 2$

| Interval | $(-\infty, 2)$ | $(2, \infty)$ |
|---|---|---|
| Sign of $f'$ | − | + |
| $f$ | Decreasing | Increasing |

Relative minimum: $(2, -44)$

**12.** $f(x) = x^4 - 12x^3$

$f'(x) = 4x^3 - 36x^2 = 4x^2(x - 9)$

Critical numbers: $x = 0, 9$

| Interval | $(-\infty, 0)$ | $(0, 9)$ | $(9, \infty)$ |
|---|---|---|---|
| Sign of $f'$ | $f' < 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion | Decreasing | Decreasing | Increasing |

Relative minimum: $(9, -2187)$

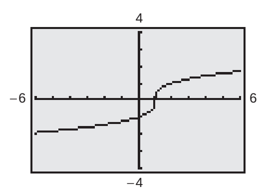
**14.** $f(t) = (t - 1)^{1/3}$

$f'(t) = \frac{1}{3}(t - 1)^{-2/3} = \frac{1}{3\sqrt[3]{(t - 1)^2}}$

Critical number: $t = 1$

| Interval | $(-\infty, 1)$ | $(1, \infty)$ |
|---|---|---|
| Sign of $f'$ | + | + |
| $f$ | Increasing | Increasing |

No relative extrema




**16.** $f(x) = x + \frac{1}{x}$

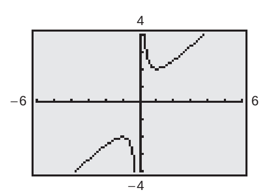
$f'(x) = 1 - \frac{1}{x^2} = 0$

Critical numbers: $x = \pm 1$

| Interval | $(-\infty, -1)$ | $(-1, 0)$ | $(0, 1)$ | $(1, \infty)$ |
|---|---|---|---|---|
| Sign of $f'$ | + | − | − | + |
| $f$ | Increasing | Decreasing | Decreasing | Increasing |

Relative maximum: $(-1, -2)$

Relative minimum: $(1, 2)$



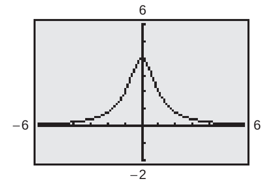
**18.** $f(x) = \dfrac{4}{x^2 + 1}$

$f'(x) = \dfrac{-8x}{(x^2 + 1)^2} = 0$

Critical number: $x = 0$

| Interval | $(-\infty, 0)$ | $(0, \infty)$ |
|---|---|---|
| Sign of $f'$ | $+$ | $-$ |
| $f$ | Increasing | Decreasing |

Relative maximum: $(0, 4)$




**20.** $f(x) = \frac{1}{3}(2x + 5), \quad [0, 5]$

$f'(x) = \frac{2}{3}$

No critical numbers

| $x$-value | Endpoint $x = 0$ | Endpoint $x = 5$ |
|---|---|---|
| $f(x)$ | $\frac{5}{3}$ | 5 |
| Conclusion | Minimum | Maximum |

**22.** $f(x) = x^2 + 2x - 4, \quad [-1, 1]$

$f'(x) = 2x + 2$

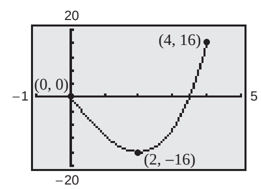
Critical number: $x = -1$ (also an endpoint)

| $x$-value | Endpoint $x = -1$ | Endpoint $x = 1$ |
|---|---|---|
| $f(x)$ | $-5$ | $-1$ |
| Conclusion | Minimum | Maximum |

**24.** $f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 0$

Critical numbers: $x = \pm 2$

| $x$-value | Endpoint $x = 0$ | Critical $x = 2$ | Endpoint $x = 4$ |
|---|---|---|---|
| $f(x)$ | 0 | $-16$ | 16 |
| Conclusion | | Minimum | Maximum |




**26.** $h(t) = \dfrac{t}{t - 2}, \quad [3, 5]$

$h'(t) = \dfrac{-2}{(t - 2)^2}$

No critical numbers

| $t$-value | Endpoint $t = 3$ | Endpoint $t = 5$ |
|---|---|---|
| $h(t)$ | 3 | $\frac{5}{3}$ |
| Conclusion | Maximum | Minimum |

**28.** $g(t) = \dfrac{t^2}{t^2 + 3}, \quad [-1, 1]$

$g'(t) = \dfrac{6t}{(t^2 + 3)^2}$

Critical number: $t = 0$

| $t$-value | Endpoint $t = -1$ | Critical $t = 0$ | Endpoint $t = 1$ |
|---|---|---|---|
| $g(t)$ | $\frac{1}{4}$ | 0 | $\frac{1}{4}$ |
| Conclusion | Maximum | Minimum | Maximum |

**30.** $g(x) = 4\left(1 + \dfrac{1}{x} + \dfrac{1}{x^2}\right) = 4(1 + x^{-1} + x^{-2}), \quad [-4, 5]$

$g'(x) = 4(-x^{-2} - 2x^{-3}) = 4\left(-\dfrac{1}{x^2} - \dfrac{2}{x^3}\right) = -4\left(\dfrac{x + 2}{x^3}\right)$

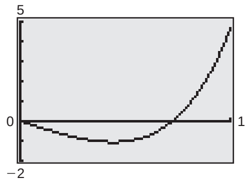
Critical number: $x = -2$
Discontinuity: $x = 0$
No maximum

| $x$-value | Endpoint $x = -4$ | Critical $x = -2$ | Discontinuity $x = 0$ | Endpoint $x = 5$ |
|---|---|---|---|---|
| $g(x)$ | 3.25 | 3 | Undefined | 4.96 |
| Conclusion | | Minimum | | |

**32.** $f(x) = 3.2x^5 + 5x^3 - 3.5x, \quad [0, 1]$

Maximum: $(1, 4.7)$

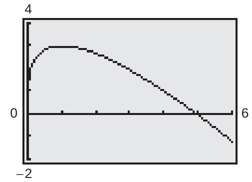
Minimum: $(0.4398, -1.0613)$




**34.** $f(x) = 4\sqrt{x} - 2x + 1, \quad [0, 6]$

Maximum: $(1, 3)$

Minimum: $(6, -1.202)$




**36.** $f(x) = \frac{8}{x+1}, \quad [0, \infty)$

$$f'(x) = \frac{-8}{(x+1)^2}$$

No critical numbers; $f$ is decreasing on $[0, \infty)$
Maximum: $f(0) = 8$

**38.** $f(x) = 8 - \frac{4x}{x^2+1}, \quad [0, \infty)$

$$f'(x) = -\frac{(x^2+1)(4) - 4x(2x)}{(x^2+1)^2} = \frac{4x^2-4}{(x^2+1)^2} = 0$$

Critical number: $x = 1$
Maximum: $f(0) = 8$
Minimum: $f(1) = 6$

**40.** $f(x) = \frac{1}{x^2+1}, \quad [0, 3]$

$$f'(x) = \frac{-2x}{(x^2+1)^2}$$

$$f''(x) = \frac{2(3x^2-1)}{(x^2+1)^3}$$

$$f'''(x) = \frac{24x(1-x^2)}{(x^2+1)^4}$$

Critical numbers for $f''$ in $[0, 3]$: $x = 0, x = 1$

| $x$-value | Endpoint $x = 0$ | Critical $x = 1$ | Endpoint $x = 3$ |
|---|---|---|---|
| $\lvert f''(x)\rvert$ | 2 | $\frac{1}{2}$ | $\frac{13}{250}$ |
| Conclusion | Maximum | | |

**42.** $f(x) = \frac{1}{x^2}, \quad [1, 2]$

$$f'(x) = -\frac{2}{x^3}$$

$$f''(x) = \frac{6}{x^4}$$

$$f'''(x) = -\frac{24}{x^5}$$

$$f^{(4)}(x) = \frac{120}{x^6}$$

$$f^{(5)}(x) = -\frac{720}{x^7}$$

No critical numbers of $f^{(4)}$

| $x$-value | Endpoint $x = 1$ | Endpoint $x = 2$ |
|---|---|---|
| $\lvert f^{(4)}(x)\rvert$ | 120 | $\frac{15}{8}$ |
| Conclusion | Maximum | |

**44.** $x = \frac{k}{p^3}, \quad p > 1$

$$8 = \frac{k}{(10)^3} \Rightarrow k = 8000$$

$$x = \frac{8000}{p^3} \Rightarrow p = \frac{20}{\sqrt[3]{x}}$$

$$C = 4x + 100$$

$$P = xp - C = x\left(\frac{20}{\sqrt[3]{x}}\right) - (4x + 100) = 20x^{2/3} - 4x - 100$$

$$\frac{dP}{dx} = \frac{40}{3}x^{-1/3} - 4 = 0 \text{ when } x = \frac{1000}{27} \approx 37 \text{ units.}$$

Since $P$ is increasing on $(1, 1000/27)$ and decreasing on $(1000/27, \infty)$, the profit is maximum when $x = 1000/27 \approx 37$ units and the price is \$6.00.

**46.** $v = k(R - r)r^2, \quad 0 \le r < R$

$= k(Rr^2 - r^3)$

$$\frac{dv}{dr} = k(2Rr - 3r^2) = kr(2R - 3r)$$

Critical numbers: $r = 0, r = \frac{2R}{3}$

| $r$-value | Endpoint $r = 0$ | Critical $r = (2R)/3$ | Endpoint $r \to R$ |
|---|---|---|---|
| $v$ | 0 | $(4kR^3)/27$ | $v \to 0$ |
| Conclusion | | Maximum | |

The maximum air velocity occurs when $r = 2R/3$.

**48.** (a) 1970: 2500 per 1000 women

(b) 1985-1990 most rapidly
1975-1980 most slowly

(c) 1970-1975 most rapidly
1980-1985 most slowly

(d) Answers will vary.

## Section 3.3 Concavity and the Second-Derivative Test

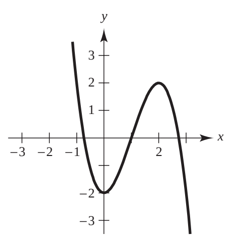
**2.** $f(x) = -x^3 + 3x^2 - 2$

$f'(x) = -3x^2 + 6x$

$f''(x) = -6x + 6$

$f''(x) = 0$ when $x = 1$.

Concave upward on $(-\infty, 1)$
Concave downward on $(1, \infty)$

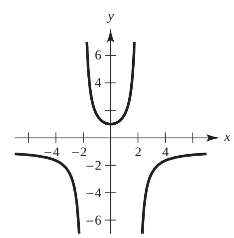



**4.** $f(x) = \frac{x^2 + 4}{4 - x^2}$

$$f'(x) = \frac{(4 - x^2)2x - (x^2 + 4)(-2x)}{(4 - x^2)^2} = \frac{16x}{(4 - x^2)^2}$$

$$f''(x) = \frac{(4 - x^2)^2 16 - 16x \cdot 2(4 - x^2)(-2x)}{(4 - x^2)^4} = \frac{-16(4 + 3x^2)}{(4 - x^2)^3}$$

$f''$ is undefined at $x = \pm 2$.
Concave upward on $(-2, 2)$
Concave downward on $(-\infty, -2)$ and $(2, \infty)$




**6.** $f(x) = \frac{x^2}{x^2 + 1}$

$$f'(x) = \frac{2x}{(x^2 + 1)^2}$$

$$f''(x) = \frac{-2(3x^2 - 1)}{(x^2 + 1)^3}$$

$f''(x) = 0$ when $x = \pm\frac{1}{\sqrt{3}} \approx \pm 0.5774$

Concave upward on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

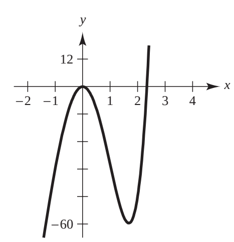
Concave downward on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$

**8.** $f(x) = x^5 + 5x^4 - 40x^2$

$f'(x) = 5x^4 + 20x^3 - 80x$

$f''(x) = 20x^3 + 60x^2 - 80$

Concave upward on $(1, \infty)$
Concave downward on $(-\infty, 1)$



**10.** $f(x) = (x - 5)^2$

$f'(x) = 2(x - 5) = 0$

Critical number: $x = 5$

$f''(x) = 2$

$f''(5) = 2 > 0$

Thus, $(5, 0)$ is a relative minimum.

**12.** $f(x) = x^4 - 4x^3 + 2$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3) = 0$

Critical numbers: $x = 0, 3$

$f''(x) = 12x^2 - 24x$

$f''(0) = 0$ Test fails

$f''(3) > 0$

$(3, -24)$ is a relative minimum.
$(0, 3)$ is not a relative extremum.

**14.** $f(x) = x + \dfrac{4}{x}$

$f'(x) = 1 - \dfrac{4}{x^2} = \dfrac{x^2 - 4}{x^2}$

Critical numbers: $x = \pm 2$

$f''(x) = \dfrac{8}{x^3}$

$f''(2) = 1 > 0$

$f''(-2) = -1 < 0$

Thus, $(2, 4)$ is a relative minimum and $(-2, -4)$ is a relative maximum.

**16.** $f(x) = \sqrt{4 - x^2}$ Domain: $-2 \le x \le 2$

$f'(x) = \dfrac{-x}{\sqrt{4 - x^2}}$

$f''(x) = \dfrac{-4}{(4 - x^2)^{3/2}}$

Critical numbers: $x = 0$

$f''(0) < 0$

$(0, 2)$ is a relative maximum.

$[(2,0)$ and $(-2, 0)$ are absolute minimums.$]$

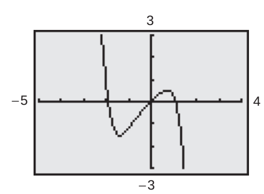
**18.** $f(x) = \dfrac{x}{x^2 - 1}$

$f'(x) = -\dfrac{1 + x^2}{(x^2 - 1)^2}$

$f''(x) = \dfrac{2x(x^2 + 3)}{(x^2 - 1)^3}$

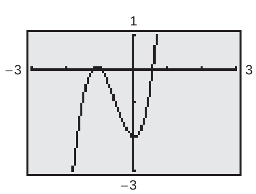
No critical numbers
No relative extrema

**20.** $f(x) = -\dfrac{1}{3}x^5 - \dfrac{1}{2}x^4 + x$




Relative maximum: $(0.6830, 0.5247)$
Relative minimum: $(-1.4128, -1.5286)$

**22.**




$(-1.11, 0.057)$ is a relative maximum.
$(0, -2)$ is a relative minimum.

**24.** $f' > 0$, (increasing)
$f'' < 0$, (concave downward)

**26.** $f' < 0$, (decreasing)
$f'' > 0$, (concave upward)

**28.** $f(x) = x(6 - x)^2 = x^3 - 12x^2 + 36x$

$f'(x) = 3x^2 - 24x + 36$

$f''(x) = 6x - 24 = 0$ when $x = 4$.

$f''(x) < 0$ on $(-\infty, 4)$

$f''(x) > 0$ on $(4, \infty)$

Thus, $(4, 16)$ is an inflection point.

**30.** $f(x) = x^4 - 18x^2 + 5$

$f'(x) = 4x^3 - 36x$

$f''(x) = 12x^2 - 36 = 12(x^2 - 3) = 0$ when $x = \pm\sqrt{3}$.

$f''(x) > 0$ on $\left(-\infty, -\sqrt{3}\right)$

$f''(x) < 0$ on $\left(-\sqrt{3}, \sqrt{3}\right)$

$f''(x) > 0$ on $\left(\sqrt{3}, \infty\right)$

Thus, $\left(-\sqrt{3}, -40\right)$ and $\left(\sqrt{3}, -40\right)$ are inflection points.

**32.** $f(x) = -4x^3 - 8x^2 + 32$

$f'(x) = -12x^2 - 16x$

$f''(x) = -24x - 16 = 0 \Longrightarrow x = \frac{-2}{3}$

Since $f''$ changes sign at $x = \frac{-2}{3}$, $\left(\frac{-2}{3}, \frac{800}{27}\right)$ is a point of inflection.

**34.** $f(t) = (1 - t)(t - 4)(t^2 - 4)$

$f'(t) = -4t^3 + 15t^2 - 20$

$f''(t) = -6t(2t - 5)$

Inflection points: $(0, 16)$, $\left(\frac{5}{2}, \frac{81}{16}\right)$

**36.** $f(x) = x^3 - 3x$

$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$

Critical numbers: $x = -1, 1$

$f''(x) = 6x$

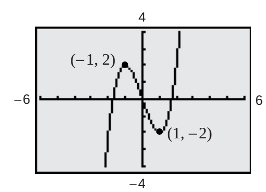
$f''(-1) = -6 < 0$

Thus $(-1, 2)$ is a relative maximum.

$f''(1) = 6 > 0$

Thus $(1, -2)$ is a relative minimum.

Point of inflection: $(0, 0)$




**38.** $f(x) = x^3 - \frac{3}{2}x^2 - 6x$

$f'(x) = 3x^2 - 3x - 6 = 3(x + 1)(x - 2)$

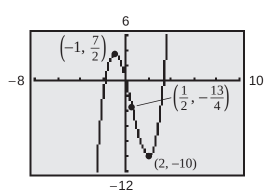
Critical numbers: $x = -1, x = 2$

$f''(x) = 6x - 3$

$f''(-1) = -9 < 0 \Longrightarrow \left(-1, \frac{7}{2}\right)$ relative maximum

$f''(2) = 9 > 0 \Longrightarrow (2, -10)$ relative minimum

Point of inflection is $\left(\frac{1}{2}, -\frac{13}{4}\right)$.




**40.** $f(x) = 2x^4 - 8x + 3$

$f'(x) = 8x^3 - 8 = 8(x^3 - 1)$

Critical number: $x = 1$

$f''(x) = 24x^2$

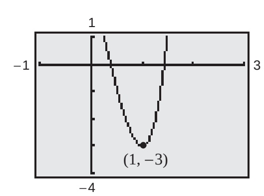
$f''(1) = 24 > 0$

Thus, $(1, -3)$ is a relative minimum.

$f''(x) = 24x^2 = 0$ when $x = 0$.

$f''(x) > 0$ on $(-\infty, 0)$ and on $(0, \infty)$

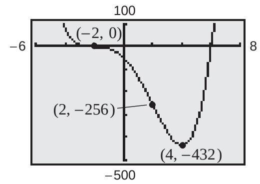
Thus, there are no inflection points.



**42.** $g(x) = (x - 6)(x + 2)^3$

$g'(x) = (x - 6)[3(x + 2)^2] + (x + 2)^3(1)$

$= (x + 2)^2[3(x - 6) + (x + 2)] = (x + 2)^2(4x - 16)$




Critical numbers: $x = -2, 4$

$g''(x) = (x + 2)^2(4) + (4x - 16)[2(x + 2)]$

$= 2(x + 2)[2(x + 2) + (4x - 16)]$

$= 2(x + 2)(6x - 12) = 12(x + 2)(x - 2)$

$g''(-2) = 0$ (Test fails)

$g''(4) = 144 > 0$

Thus, $(4, -432)$ is a relative minimum. By the First-Derivative Test, $(-2, 0)$ is not a relative extremum.

$g''(x) = 12(x + 2)(x - 2) = 0$ when $x = \pm 2$.

$g''(x) > 0$ on $(-\infty, -2)$

$g''(x) < 0$ on $(-2, 2)$

$g''(x) > 0$ on $(2, \infty)$

Thus, $(-2, 0)$ and $(2, -256)$ are points of inflection.

**44.** $g(x) = x\sqrt{9 - x}$

$g'(x) = \dfrac{18 - 3x}{2\sqrt{9 - x}}$

$g''(x) = \dfrac{3(x - 12)}{4(9 - x)^{3/2}}$

Critical number: $x = 6$

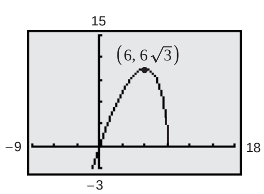
$g''(6) < 0$

Relative maximum: $\left(6, 6\sqrt{3}\right) \approx (6, 10.3923)$

$g'' < 0$ for all $x$ in domain.

Concave downward on $(-\infty, 9)$.

No points of inflection




**46.** $f(x) = \dfrac{2}{x^2 - 1}$

$f'(x) = \dfrac{-4x}{(x^2 - 1)^2}$

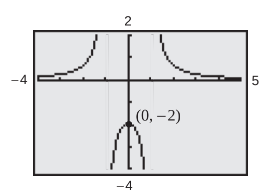
$f''(x) = \dfrac{4(3x^2 + 1)}{(x^2 - 1)^3}$

Critical number: $x = 0$

$f''(0) < 0$

$(0, -2)$ is a relative maximum.

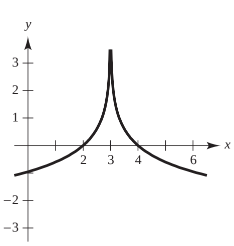
No points of inflection

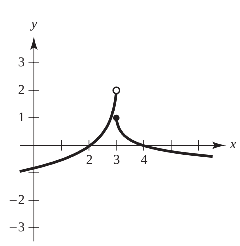



**48.**

| *Function* | *First Derivative* | *Second Derivative* |
|---|---|---|
| $f(2) = 0$ | $f'(x) > 0, \quad x < 3$ | $f''(x) > 0, \quad x \neq 3$ |
| $f(4) = 0$ | $f'(3)$ is undefined. | |
| | $f'(x) < 0, \quad x > 3$ | |

The function has $x$-intercepts at $(2, 0)$ and $(4, 0)$. On $(-\infty, 3)$, $f$ is increasing and on $(3, \infty)$, $f$ is decreasing. $f$ has either a relative maximum at $x = 3$ or a discontinuity at $x = 3$. Also, $f$ is concave upward on $(-\infty, 3)$ and $(3, \infty)$.



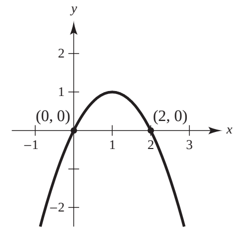



**50.** (a) $f'$ is positive on $(0, 2)$ where $f$ is increasing.

(b) $f'$ is negative on $(-\infty, 0)$ and $(2, \infty)$ where $f$ is decreasing.

(c) $f'$ is increasing on $(-\infty, 1)$ where $f$ is concave upward.

(d) $f'$ is decreasing on $(1, \infty)$ where $f$ is concave downward.




**52.** $R' = -\frac{4}{9}(3x^2 - 18x)$

$R'' = -\frac{4}{9}(6x - 18) = 0$ when $x = 3$.

$R'' > 0$ on $(0, 3)$

$R'' < 0$ on $(3, 5)$

Since $(3, 36)$ is a point of inflection, it is the point of diminishing returns.

**54.** $C = 0.002x^3 + 20x + 500$

$\overline{C} = 0.002x^2 + 20 + \frac{500}{x}$

$\overline{C}' = 0.004x - \frac{500}{x^2} = 0$

$x^3 = \frac{500}{0.004}$

$x = 50$ units

**56.** $N(t) = \frac{20t^2}{4 + t^2}, \quad 0 \le t \le 4$

We need to determine when $N'(t)$ is greatest.

$$\begin{aligned} N'(t) &= \frac{(4 + t^2)(40t) - (20t^2)(2t)}{(4 + t^2)^2} \\ &= 160t(4 + t^2)^{-2} \\ N''(t) &= 160\{t[-2(4 + t^2)^{-3}(2t)] + (4 + t^2)^{-2}(1)\} \\ &= 160(4 + t^2)^{-3}[-4t^2 + (4 + t^2)] \\ &= \frac{160(4 - 3t^2)}{(4 + t^2)^3} \\ &= 0 \text{ when } t = \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}. \end{aligned}$$

The student is assembling components at the greatest rate when $t = (2\sqrt{3})/3$ or at approximately 8:09 P.M.

**58.** $S(t) = \frac{500{,}000t^2}{36 + t^2}$

$$\begin{aligned} S'(t) &= 500{,}000\left[\frac{(36 + t^2)(2t) - (t^2)(2t)}{(36 + t^2)^2}\right] \\ &= 36{,}000{,}000t(36 + t^2)^{-2} \\ S''(t) &= 36{,}000{,}000t[-2(36 + t^2)^{-3}(2t) + (36 + t^2)^{-2}(1)] \\ &= 36{,}000{,}000(36 + t^2)^{-3}[-4t^2 + (36 + t^2)] \\ &= \frac{36{,}000{,}000(36 - 3t^2)}{(36 + t^2)^3} \\ &= 0 \text{ when } t = \pm\sqrt{12} = \pm 2\sqrt{3}. \end{aligned}$$

The sales of this new product is increasing at the greatest rate when $t = 2\sqrt{3} \approx 3.464$ years.

**60.** $f(x) = -\frac{1}{20}x^5 - \frac{1}{12}x^2 - \frac{1}{3}x + 1, \ [-2, 2]$

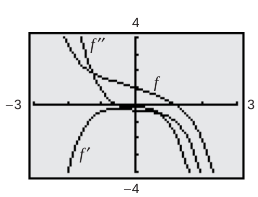
$f'(x) = -\frac{1}{4}x^4 - \frac{1}{6}x - \frac{1}{3}$

$f''(x) = -x^3 - \frac{1}{6}$

Minimum: $(2, -1.6)$

Maximum: $(-2, 2.93)$

Point of inflection: $(-0.55, 1.16)$



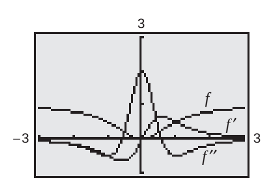

**62.** $f(x) = \frac{x^2}{x^2 + 1}, \quad [-3, 3]$

$f'(x) = \frac{(x^2 + 1)(2x) - x^2(2x)}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}$

$f''(x) = \frac{2(1 - 3x^2)}{(x^2 + 1)^3}$

Relative minimum: $(0, 0)$

Inflection points: $\left(\frac{1}{\sqrt{3}}, \frac{1}{4}\right), \left(-\frac{1}{\sqrt{3}}, \frac{1}{4}\right)$



**64.** (a) $S'' > 0$

(b) $S' > 0$ and $S'' > 0$

(c) $S'' = 0$

(d) $S' = 0$

(e) $S' < 0$ and $S'' > 0$

(f) $S' = 0$ and $S'' > 0$

**66.** Answers will vary.

## Section 3.4 Optimization Problems

**2.** Let $x$ be the first number and $y$ be the second number. Then $x + y = S$ and $y = S - x$. Thus, the product of $x$ and $y$ is given by the following.

$$P = xy = x(S - x) = Sx - x^2$$

$$P' = S - 2x$$

$P' = 0$ when $x = S/2$. Since $P''(S/2) = -2 < 0$, the product is maximum when $x = y = S/2$.

**4.** Let $x$ be the first number and $y$ be the second number. Then $x + 2y = 100$ and $x = 100 - 2y$. Thus, the product of $x$ and $y$ is given by the following.

$$P = xy = (100 - 2y)y = 100y - 2y^2$$

$$P' = 100 - 4y$$

$P' = 0$ when $y = 25$. Since $P''(25) = -4 < 0$, the product is maximum when $x = 100 - 2(25) = 50$ and $y = 25$.

**6.** Let $x$ be the first number and $y$ be the second number. Then $xy = 192$ and $y = 192/x$. The sum is given by the following.

$$S = x + 3y = x + 3\left(\frac{192}{x}\right) = x + 576x^{-1}$$

$$S' = 1 - 576x^{-2} = \frac{x^2 - 576}{x^2}$$

Critical number: $x = 24$

$$S'' = \frac{1152}{x^3}$$

Since $S''(24) > 0$, the sum is minimum when $x = 24$ and $y = 192/24 = 8$.

**8.** Let $x$ be the first number and $y$ be the second number. Then $x - y = 50$ and $x = 50 + y$. The product is given by the following.

$$P = xy = (50 + y)y = 50y + y^2$$

$$P' = 50 + 2y$$

$P' = 0$ when $y = -25$. Since $P''(-25) = 2 > 0$, the product is minimum when $x = 50 + (-25) = 25$ and $y = -25$.

**10.** Let $x$ be the length and $y$ be the width of the rectangle. Then $2x + 2y = P$ and $y = \frac{1}{2}(P - 2x)$. The area is given by the following.

$$A = xy = x\tfrac{1}{2}(P - 2x) = \tfrac{1}{2}Px - x^2$$

$$A' = \tfrac{1}{2}P - 2x$$

$A' = 0$ when $x = P/4$. Since $A'' = -2 < 0$, $A$ is a maximum when $x = P/4$ and $y = P/4$. This is a square!

**12.** Let $x$ and $y$ be the length and width of the rectangle. The area is $xy = A$ and $y = A/x$. The perimeter is given by the following.

$$P = 2x + 2y = 2x + 2\left(\frac{A}{x}\right) = 2x + 2Ax^{-1}$$

$$P' = 2 - 2Ax^{-2} = 2 - \frac{2A}{x^2} = \frac{2x^2 - 2A}{x^2}$$

$P' = 0$ when $x^2 = A$, or $x = \sqrt{A}$.

$P''(x) = 4Ax^{-3} > 0 \implies$ this is a minimum.

Hence, $x = \sqrt{A}$, $y = \sqrt{A}$ (a square!)

**14.** Let $x$ and $y$ be the lengths shown in the figure. Then $xy = 180{,}000$ and $y = 180{,}000/x$. The perimeter is given by the following.

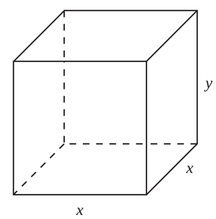
$$P = 2x + y = 2x + \frac{180{,}000}{x}$$

$$P' = 2 - \frac{180{,}000}{x^2} = \frac{2(x^2 - 90{,}000)}{x^2}$$

$P' = 0$ when $x = 300$. Since $P'' = \dfrac{360{,}000}{x^3}$ and $P''(300) > 0$,

$P$ is minimum when $x = 300$ meters and $y = \dfrac{180{,}000}{300} = 600$ meters.

**16.** Let $x$ be the sides of the base, and $y$ the height.




Surface area $= 337.5 = 2x^2 + 4xy \implies y = \dfrac{1}{4x}[337.5 - 2x^2]$

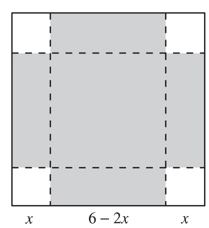
$$V = x^2y = x^2\left[\frac{1}{4x}(337.5 - 2x^2)\right] = \frac{675}{8}x - \frac{x^3}{2}$$

$$V'(x) = \frac{675}{8} - \frac{3x^2}{2} = 0 \implies x = 7.5 \implies y = 7.5$$

$V''(x) = -3x < 0 \implies x = 7.5$ is a maximum.

Dimensions: $7.5 \times 7.5 \times 7.5$

**18.** Let $x$ be the length shown in the figure. Then the volume of the box is given by the following.

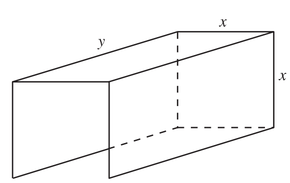



$$V = x(6 - 2x)^2, \quad 0 < x < 3$$

$$V' = 12(x - 1)(x - 3)$$

$V' = 0$ when $x = 3$ and $x = 1$. Since $V = 0$ when $x = 3$ and $V = 16$ when $x = 1$, we conclude that the volume is maximum when $x = 1$. The corresponding volume is $V = 16$ cubic inches.

**20.** Let $x$ and $y$ be the lengths shown in the figure. Then $x^2y = 250/3$ and $y = 250/3x^2$. The surface area of the enclosure is given by the following.




$$A = 3xy + x^2 = 3x\left(\frac{250}{3x^2}\right) + x^2 = \frac{250}{x} + x^2$$

$$A' = -\frac{250}{x^2} + 2x = \frac{2x^3 - 250}{x^2}$$

$A' = 0$ when $x = 5$. The surface area is minimum when $x = 5$ meters and

$$y = \frac{250}{3(5)^2} = \frac{10}{3} \text{ meters.}$$

**22.** $A = (x + 3)(y + 3)$

$$36 = xy \Longrightarrow y = \frac{36}{x}$$

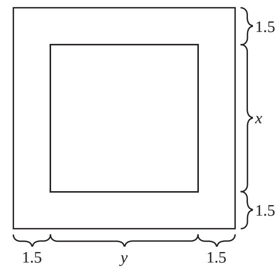
$$A = (x + 3)\left(\frac{36}{x} + 3\right)$$

$$= 36 + \frac{108}{x} + 3x + 9$$

$$\frac{dA}{dx} = -\frac{108}{x^2} + 3 = 0$$

$$x^2 = 36$$

$$x = 6, y = 6$$




By the First-Derivative Test, $A$ is a minimum when $x = y = 6$. Thus, length = width = 9 inches.

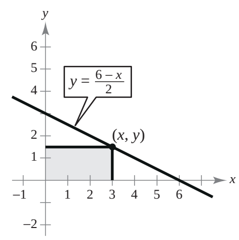
**24.** The area of the rectangle is

$$A = xy = x\left(\frac{6 - x}{2}\right) = \frac{1}{2}(6x - x^2)$$

$$A' = \frac{1}{2}(6 - 2x).$$

$A' = 0$ when $x = 3$. Thus, $A$ is maximum when $x = 3$ and

$$y = \frac{6 - 3}{2} = \frac{3}{2}.$$




**26.** The area is given by the following.

$$A = 2xy = 2x\sqrt{25 - x^2}$$

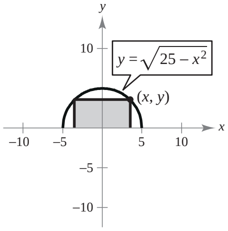
$$A' = 2\left(\frac{25 - 2x^2}{\sqrt{25 - x^2}}\right)$$

$A' = 0$ when $x = 5/\sqrt{2}$. Thus, $A$ is maximum when the length is

$$2x = \frac{10}{\sqrt{2}} \approx 7.07$$

and the width is

$$y = \sqrt{25 - \left(\frac{5}{\sqrt{2}}\right)^2} = \frac{5}{\sqrt{2}} \approx 3.54.$$




**28.** The volume of the cylinder is

$$V = \pi r^2 h = 12(1.80469) \approx 21.66$$

which implies that $h = 21.66/\pi r^2$. The surface area of the cylinder is

$$S = 2\pi r^2 + 2\pi r h$$

$$= 2\pi r^2 + 2\pi r\left(\frac{21.66}{\pi r^2}\right)$$

$$= 2\left(\pi r^2 + \frac{21.66}{r}\right)$$

$$S' = 2\left(2\pi r - \frac{21.66}{r^2}\right).$$

$S' = 0$ when $2\pi r^3 - 21.66 = 0$, which implies that

$$r = \sqrt[3]{\frac{21.66}{2\pi}} \approx 1.51 \text{ inches}$$

$$h = \frac{21.66}{\pi(1.51)^2} \approx 3.02 \text{ inches.}$$

(Note that in the solution, $h = 2r$.)

**30.** $V = \frac{1}{3}\pi x^2 h = \frac{1}{3}\pi x^2\left(r + \sqrt{r^2 - x^2}\right)$ (see figure)

$$\frac{dV}{dx} = \frac{1}{3}\pi\left[\frac{-x^3}{\sqrt{r^2 - x^2}} + 2x\left(r + \sqrt{r^2 - x^2}\right) - \frac{\pi x}{3\sqrt{r^2 - x^2}}\left(2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2\right) = 0\right.$$

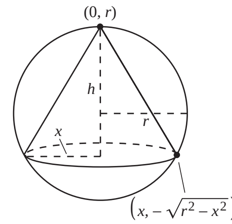
$$2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2 = 0$$

$$2r\sqrt{r^2 - x^2} = 3x^2 - 2r^2$$

$$4r^2(r^2 - x^2) = 9x^4 - 12x^2r^2 + 4r^4$$

$$0 = 9x^4 - 8x^2r^2 = x^2(9x^2 - 8r^2)$$

$$x = 0, \frac{2\sqrt{2}r}{3}$$




By the First Derivative Test, the volume is a maximum when

$$x = \frac{2\sqrt{2}r}{3} \text{ and } h = r + \sqrt{r^2 - x^2} = \frac{4r}{3}.$$

Thus, the maximum volume is

$$V = \frac{1}{3}\pi\left(\frac{8r^2}{9}\right)\left(\frac{4r}{3}\right) = \frac{32\pi r^2}{81} \text{ cubic units.}$$

**32.** The distance between a point $(x, y)$ on the graph and the point $(2, 1/2)$ is

$$d = \sqrt{(x - 2)^2 + \left(y - \frac{1}{2}\right)^2} = \sqrt{(x - 2)^2 + \left(x^2 - \frac{1}{2}\right)^2}$$

and we can minimize $d$ by minimizing its square $L = d^2$.

$$L = (x - 2)^2 + \left(x^2 - \frac{1}{2}\right)^2 = x^4 - 4x + \frac{17}{4}$$

$$L' = 4x^3 - 4$$

$L' = 0$ when $x = 1$ and $y = (1)^2 = 1$. Thus, the point nearest $(2, 1/2)$ is $(1, 1)$.

**34.** The volume is $\frac{4}{3}\pi r^3 + \pi r^2 h = 12$, thus $h = \frac{12}{\pi r^2} - \frac{4}{3}r$.
The surface area is given by the following.

$$A = 4\pi r^2 + 2\pi r h$$

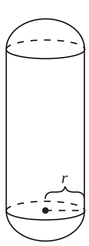
$$= 4\pi r^2 + 2\pi r\left(\frac{12}{\pi r^2} - \frac{4}{3}r\right)$$

$$= 4\pi r^2 + \frac{24}{r} - \frac{8}{3}\pi r^2$$

$$= \frac{4}{3}\pi r^2 + \frac{24}{r}$$

$$A' = \frac{8}{3}\pi r - \frac{24}{r^2} = \frac{8(\pi r^3 - 9)}{3r^2}$$

$A' = 0$ when $r = \sqrt[3]{9/\pi} \approx 1.42$ in.




**36.** The perimeter is given by $3x + 4y = 10$. Thus, $y = \frac{1}{4}(10 - 3x)$. The total area is given by the following.

$$A = \frac{1}{2}bh + y^2 = \frac{1}{2}x\left(\frac{\sqrt{3}x}{2}\right) + \left[\frac{1}{4}(10 - 3x)\right]^2$$

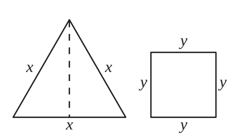
$$= \frac{\sqrt{3}}{4}x^2 + \frac{1}{16}(100 - 60x + 9x^2)$$

$$A' = \frac{\sqrt{3}}{2}x + \frac{1}{16}(-60 + 18x)$$

$$= x\left(\frac{\sqrt{3}}{2} + \frac{9}{8}\right) - \frac{15}{4}$$

$A' = 0$ when $x = \frac{30}{4\sqrt{3} + 9}$ and

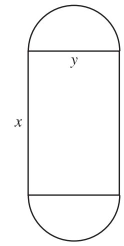
$$y = \frac{1}{4}\left[10 - 3\left(\frac{30}{4\sqrt{3} + 9}\right)\right] = \frac{10\sqrt{3}}{4\sqrt{3} + 9}.$$



**38.** Let $x$ and $y$ be the length and width of the rectangle. The radius of the semicircle is $r = y/2$, and the perimeter is

$$200 = 2x + 2\pi r = 2x + 2\pi\left(\frac{y}{2}\right) = 2x + \pi y$$

which implies that $y = (200 - 2x)/\pi$. The area of the rectangle is given by the following.



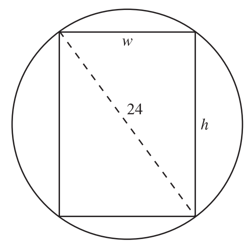

$$A = xy = x\left[\frac{200 - 2x}{\pi}\right] = \frac{2}{\pi}(100x - x^2)$$

$$A' = \frac{2}{\pi}(100 - 2x)$$

$A' = 0$ when $x = 50$. Thus, $A$ is maximum when $x = 50$ meters and

$$y = \frac{200 - 2(50)}{\pi} = \frac{100}{\pi} \text{ meters.}$$

**40.** Since $h^2 + w^2 = 24^2$, we have $h^2 = 24^2 - w^2$.




$$S = kh^2w = k(24^2 - w^2)w = k(576w - w^3)$$

$$S' = k(576 - 3w^2)$$

$S' = 0$ when $w = \sqrt{192} = 8\sqrt{3}$. Thus, $S$ is maximum when

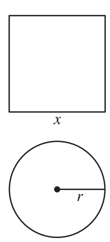
$$h^2 = 24^2 - 192 = 384 \Rightarrow h = \sqrt{384} = 8\sqrt{6}.$$

The dimensions are $w = 8\sqrt{3} \approx 13.856$ inches and $h = 8\sqrt{6} \approx 19.596$ inches.

**42.** (a) Area $= x^2 + \pi r^2$

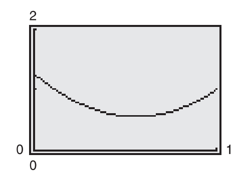
$$4 = 4x + 2\pi r \Rightarrow r = \frac{2 - 2x}{\pi}$$

$$A(x) = x^2 + \pi\left(\frac{2 - 2x}{\pi}\right)^2 = x^2 + \frac{(2 - 2x)^2}{\pi}$$




(b) Domain: $0 \le x \le 1$

(c)




(d) $A$ is minimum if $x = 0.5601$ and $r = 0.28$;

(e) $A$ is maximum if $x = 0$ and $r = 2/\pi$ (all the wire for the circle).

## Section 3.5 Business and Economics Applications

**2.** $R = 48x^2 - 0.02x^3$

$R' = 96x - 0.06x^2 = x(96 - 0.06x)$

$R' = 0$ when $x = 0$ or $x = 1600$.

Thus, $R$ is maximum when $x = 1600$ units.

**4.** $R = 30x^{2/3} - 2x$

$$R' = 20x^{-1/3} - 2 = \frac{20}{\sqrt[3]{x}} - 2$$

$R' = 0$ when $x = 1000$.

Thus, $R$ is maximum when $x = 1000$ units.

**6.** $\overline{C} = 0.001x^2 + 5 + \dfrac{250}{x}$

$\overline{C}' = 0.002x - \dfrac{250}{x^2} = \dfrac{0.002x^3 - 250}{x^2}$

$\overline{C}' = 0$ when $x = 50$.

Thus, $\overline{C}$ is minimum when $x = 50$ units.

**8.** $\overline{C} = 0.02x^2 + 55x + \dfrac{1250}{x}$

$\overline{C}' = 0.04x + 55 - \dfrac{1250}{x^2} = 0$

$\dfrac{0.04x^3 + 55x^2 - 1250}{x^2} = 0 \Longrightarrow x \approx 5$ units

**10.** $P = xp - C = x\left(\dfrac{60}{\sqrt{x}}\right) - (0.5x + 600)$

$= 60x^{1/2} - 0.5x - 600$

$P' = 30x^{-1/2} - 0.5$

$P' = 0$ when $\dfrac{30}{\sqrt{x}} = \dfrac{1}{2} \Longrightarrow \sqrt{x} = 60 \Longrightarrow x = 3600$

By the First Derivative Test, this is a maximum.

$x = 3600,\ p = \dfrac{60}{\sqrt{3600}} = 1$ dollar

**12.** $P = xp - C = x\left(50 - \dfrac{\sqrt{x}}{10}\right) - (35x + 500)$

$= 15x - \dfrac{1}{10}x^{3/2} - 500$

$P' = 15 - \dfrac{3}{20}x^{1/2}$

$P' = 0$ when $15 = \dfrac{3}{20}\sqrt{x}$

$100 = \sqrt{x}$

$10{,}000 = x.$

By the First-Derivative Test, $P$ is a maximum when $x = 10{,}000$ units and $p = 50 - \dfrac{\sqrt{10{,}000}}{10} = \$40.$

**14.** $C = x^3 - 6x^2 + 13x$

$\overline{C} = x^2 - 6x + 13$

$\overline{C}' = 2x - 6$

$\overline{C}' = 0$ when $x = 3$.

Thus, the average cost is minimum when $x = 3$ units and $\overline{C}(3) = \$4$ per unit.

$C' =$ marginal cost $= 3x^2 - 12x + 13$

$C'(3) = 4 = \overline{C}(3)$

**16.** (a) $P = xp - C = x\left(100 - \dfrac{1}{2}x^2\right) - (50x + 37.5) = -\dfrac{1}{2}x^3 + 50x - 37.50$

$P' = -\dfrac{3}{2}x^2 + 50$

$P' = 0$ when $x = \dfrac{10}{\sqrt{3}}$ units and $p = 100 - \dfrac{1}{2}\left(\dfrac{10}{\sqrt{3}}\right)^2 = \dfrac{250}{3} \approx \$83.33.$

(b) $\overline{C} = 50 + \dfrac{37.5}{x}$. When $x = \dfrac{10}{\sqrt{3}}$, the average price is $\overline{C}\left(\dfrac{10}{\sqrt{3}}\right) = 50 + \dfrac{37.5}{(10/\sqrt{3})} \approx \$56.50.$

**18.** $P = -\frac{1}{10}s^3 + 6s^2 + 400$

$P' = -\frac{3}{10}s^2 + 12s = s\left(-\frac{3}{10}s + 12\right)$

$P' = 0$ when $s = 0$ or $s = 40$.

$P'' = -\frac{3}{5}s + 12$

$P''(0) = 12 > 0 \Longrightarrow$ Minimum

$P''(40) = -12 < 0 \Longrightarrow$ Maximum

$P'' = -\frac{3}{5}s + 12 = 0$ when $s = 20$.

The maximum profit occurs when $s = 40$ (or \$40,000) and the point of diminishing returns occurs at $s = 20$ (or \$20,000). The point is (20,000, 2,000,000).

**20.** Let $x =$ the number of \$40 increases in rent.

Rent $= 580 + 40x$

Number of apartments $= 50 - x$

Cost per apartment $= 45$

Profit $= P =$ (Rent)(Number of apartments) $-$ (Cost)(Number of apartments)

$= (580 + 40x)(50 - x) - 45(50 - x)$

$= -40x^2 + 1465 + 26{,}750$

$P'(x) = -80x + 1465$

$P'0 =$ when $x = 18.3125$.

If $x = 18$, $P = 40, 160$

If $x = 19$, $P = 40, 145$

So, $x = 18$ and the rent is $580 + 40(18) = \$1300$.

**22.** $A = ki^2$

$P = 0.12A - i(A) = ki^2(0.12 - i)$

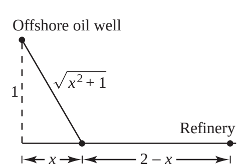
$P' = ki^2(-1) + (0.12 - i)(2ki) = 0.24ki - 3ki^2 = ki(0.24 - 3i)$

$P' = 0$ when $i = 0$ and $i = 0.08$. The profit is maximum when $i = 8\%$.

**24.** Let $k =$ cost per mile to run the line overland. Then

$C = 2k\sqrt{x^2 + 1} + k(2 - x) = k\left[2\sqrt{x^2 + 1} + (2 - x)\right]$

$C' = k\left[\frac{2x}{\sqrt{x^2 + 1}} - 1\right].$




$C' = 0$ when $\frac{2x}{\sqrt{x^2 + 1}} = 1$

$2x = \sqrt{x^2 + 1}$

$4x^2 = x^2 + 1$

$3x^2 = 1$

$x = \sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{3}$ mile.

**26.** $C = \left(\frac{v^2 + 360}{720} + 8\right)\left(\frac{110}{v}\right)$

$$= \frac{110}{720}\left(v + \frac{360}{v}\right) + \frac{880}{v}$$

$$C' = \frac{11}{72}\left(1 - \frac{360}{v^2}\right) - \frac{880}{v^2} = \frac{11}{72} - \frac{935}{v^2}$$

$C' = 0$ when $11v^2 = 72.935$

$v \approx 78.23$ mph.

**28.** Since $dp/dx = -0.03$, the price elasticity of demand is

$$\eta = \frac{p/x}{dp/dx} = \frac{(5 - 0.03x)/x}{-0.03} = 1 - \frac{5}{0.03x}.$$

When $x = 100$, we have

$$\eta = 1 - \frac{5}{0.03(100)} = -\frac{2}{3}.$$

Since $|\eta(100)| = \frac{2}{3} < 1$, the demand is inelastic.

**30.** Since $\frac{dp}{dx} = -\frac{500}{(x + 2)^2}$, the price elasticity of demand is

$$\eta = \frac{p/x}{dp/dx} = \frac{500}{x(x + 2)} \cdot \frac{(x + 2)^2}{-500} = -\frac{x + 2}{x}.$$

When $x = 23$, we have

$$\eta = -\frac{23 + 2}{23} = -\frac{25}{23}.$$

Since $|\eta(23)| = \frac{25}{23} > 1$, the demand is elastic.

**32.** Since $\frac{dp}{dx} = -\frac{0.1}{\sqrt{0.2x}}$, the price elasticity of demand is

$$\eta = \frac{p/x}{dp/dx} = \frac{100 - \sqrt{0.2x}}{x} \cdot \frac{\sqrt{0.2x}}{-0.1}$$

$$= 2 - \frac{1000\sqrt{0.2x}}{x}.$$

When $x = 125$, we have

$$\eta = 2 - \frac{1000\sqrt{0.2(125)}}{125} = -38.$$

Since $|\eta(125)| = 38 > 1$, the demand is elastic.

**34.** (a) $p^3 + x^3 = 9$

$$3p^2\frac{dp}{dx} + 3x^2 = 0$$

$$\frac{dp}{dx} = -\frac{x^2}{p^2}$$

$$\eta = \frac{p/x}{-x^2/p^2} = -\frac{p^3}{x^3}$$

When $x = 2$, $p = 1$, and $\eta = -\frac{(1)^3}{(2)^3} = -\frac{1}{8}$.

(b) $R = xp = x\sqrt[3]{9 - x^3}$

$$R' = x\left[\frac{1}{3}(9 - x^3)^{-2/3}(-3x^2)\right] + (9 - x^3)^{1/3}(1)$$

$$= (9 - x^3)^{-2/3}[-x^3 + (9 - x^3)] = \frac{9 - 2x^3}{(9 - x^3)^{2/3}}$$

$R' = 0$ when $x = \sqrt[3]{9/2}$ and $p = \sqrt[3]{9/2}$.

(c) $$\eta\left(\sqrt[3]{9/2}\right) = -\frac{\left(\sqrt[3]{9/2}\right)^3}{\left(\sqrt[3]{9/2}\right)^3} = -1$$

$$\left|\eta\left(\sqrt[3]{9/2}\right)\right| = 1$$

**36.** $C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \geq 1$

$$C' = \frac{3000x^3 - 40{,}000x^2 - 2{,}400{,}000x - 36{,}000{,}000}{x^5 + 60x^4 + 900x^3}$$

Using the root feature, $C' = 0$ for $x \approx 40.45$

Minimum obtained with order size of 4045.

**38.** $x = 800 - 40p$

$$p = 20 - \frac{x}{40}$$

$$\frac{dp}{dx} = \frac{-1}{40}$$

When $p = 5$, $x = 600$; $\eta = \frac{p/x}{dp/dx}$. Since $|\eta| < 1$, the demand is inelastic. No, revenue cannot be increased by lowering the price.

**40.** (a) $S = 201.556t^2 - 502.29t + 2622.8 + \dfrac{9286}{t}, \quad 4 \le t \le 13$

$(t = 4$ corresponds to 1994.$)$

$S' = 403.112t - 502.29 - \dfrac{9286}{t^2} > 0$

$S'' = 403.112t + \dfrac{18{,}572}{t^3} \neq 0$

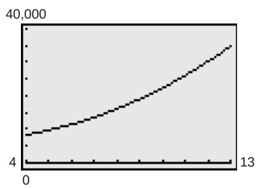
$S$ is increasing most rapidly at $t = 13$ (2003).

(b) $S$ is increasing most slowly at $t = 4$ (1994).

(c) 2003: $S'(13) \approx \$4683$ millions per year per year

1994: $S'(4) \approx \$530$ millions per year per year

(d)




**42.** (a) Demand function

(b) Cost function

(c) Revenue function

(d) Profit function

## Section 3.6 Asymptotes

**2.** A horizontal asymptote occurs at $y = 0$ since $\displaystyle\lim_{x\to\infty} \frac{4}{(x-2)^3} = 0$ and $\displaystyle\lim_{x\to-\infty} \frac{4}{(x-2)^3} = 0$.

A vertical asymptote occurs at $x = 2$ since $\displaystyle\lim_{x\to2^-} \frac{4}{(x-2)^3} = -\infty$ and $\displaystyle\lim_{x\to2^+} \frac{4}{(x-2)^3} = \infty$.

**4.** A horizontal asymptote occurs at $y = -1$ since $\displaystyle\lim_{x\to\infty} \frac{2+x}{1-x} = -1$ and $\displaystyle\lim_{x\to-\infty} \frac{2+x}{1-x} = -1$.

A vertical asymptote occurs at $x = 1$ since $\displaystyle\lim_{x\to1^-} \frac{2+x}{1-x} = \infty$ and $\displaystyle\lim_{x\to1^+} \frac{2+x}{1-x} = -\infty$.

**6.** A horizontal asymptote occurs at $y = 0$ since $\displaystyle\lim_{x\to\infty} \frac{-4x}{x^2+4} = 0$ and $\displaystyle\lim_{x\to-\infty} \frac{-4x}{x^2+4} = 0$.

The graph has no vertical asymptotes.

**8.** A horizontal asymptote occurs at $y = 0$ since $\displaystyle\lim_{x\to\infty} \frac{x^2+1}{x^3-8} = 0$ and $\displaystyle\lim_{x\to-\infty} \frac{x^2+1}{x^3-8} = 0$.

A vertical asymptote occurs at $x = 2$ since $\displaystyle\lim_{x\to2^-} \frac{x^2+1}{x^3-8} = -\infty$ and $\displaystyle\lim_{x\to2^+} \frac{x^2+1}{x^3-8} = \infty$.

**10.** The graph of $f$ has horizontal asymptote at $y = \pm 2$. It has no vertical asymptote and it matches graph (b).

**12.** The graph of $f$ has a horizontal asymptote at $y = 2$. It has no vertical asymptote and it matches graph (a) since it has a $y$-intercept at $(0, 2)$.

**14.** The graph of $f$ has a horizontal asymptote at $y = 2$. It has no vertical asymptote. The $y$-intercept is $(0, 5)$. Thus, it matches graph (d).

**16.** $\lim_{x\to -2^-} \frac{1}{x+2} = -\infty$

**18.** $\lim_{x\to 1^+} \frac{2+x}{1-x} = -\infty$

**20.** $\lim_{x\to 4} \frac{x^2}{x^2+16} = \frac{1}{2}$

**22.** $\lim_{x\to 0^-} \left(x^2 - \frac{1}{x}\right) = 0 - (-\infty) = \infty$

**24.** $\lim_{x\to\infty} \frac{5x^3+1}{10x^3-3x^2+7} = \frac{5}{10} = \frac{1}{2}$

**26.** $\lim_{x\to\infty} \frac{2x^{10}-1}{10x^{11}-3} = 0$

**28.** $\lim_{x\to\infty} \frac{x^3-2x^2+3x+1}{x^2-3x+2} = \infty$

**30.** $\lim_{x\to\infty} \left(2 - \frac{1}{x^3}\right) = 2 - 0 = 2$

**32.** $\lim_{x\to\infty} \left(\frac{2x^2}{x-1} + \frac{3x}{x+1}\right) = \infty + 3 = \infty$

**34.**

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 1 | 5.132 | 50.126 | 500.125 | 5000.126 | 50,000.2 | 500,000 |

$\lim_{x\to\infty} \left[x^2 - x\sqrt{x(x-1)}\right] = \infty$

**36.** $f(x) = \frac{3x^2}{0.1x^2+1}$

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 2.7273 | 27.2727 | 29.9700 | 29.9997 | 30 | 30 | 30 |

$\lim_{x\to\infty} f(x) = 30$

**38.**

| $x$ | $-10^6$ | $-10^4$ | $-10^2$ | $10^0$ | $10^2$ | $10^4$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $-2{,}000{,}000.5$ | $-20{,}000.5$ | $-200.5$ | 1 | 0.501 | 0.500 | 0.500 |

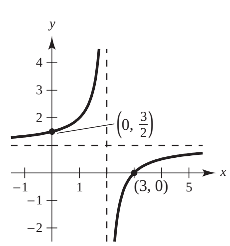
$\lim_{x\to\infty} \left[x - \sqrt{x(x-1)}\right] = 0.5,\quad \lim_{x\to-\infty} \left[x - \sqrt{x(x-1)}\right] = -\infty$

**40.** Intercepts: $(3, 0), \left(0, \frac{3}{2}\right)$

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 2$

No relative extrema



**42.** $f(x) = \dfrac{x}{x^2 + 4}$

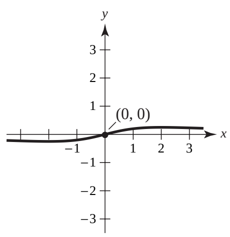
$f'(x) = \dfrac{4 - x^2}{(x^2 + 4)^2} = 0 \implies x = \pm 2$

Intercept: $(0, 0)$

Relative maximum: $\left(2, \frac{1}{4}\right)$

Relative minimum: $\left(2, -\frac{1}{4}\right)$

Horizontal asymptote: $y = 0$



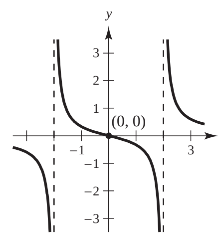

**44.** Intercept: $(0, 0)$

Horizontal asymptote: $y = 0$

Vertical asymptotes: $x = \pm 2$

$g'(x) = \dfrac{-x^2 - 4}{(x^2 - 4)^2}$

No relative extrema




**46.** $y = \dfrac{4}{x^2}$

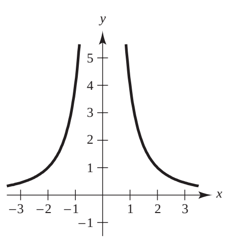
No intercepts

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = 0$

$y' = -\dfrac{8}{x^3}$

No relative extrema



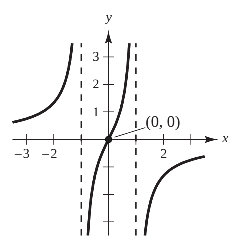
**48.** Intercept: $(0, 0)$

Horizontal asymptote: $y = 0$

Vertical asymptotes: $x = \pm 1$

$y' = -\dfrac{2(x^2 + 1)}{(1 - x^2)^2}$

No relative extrema



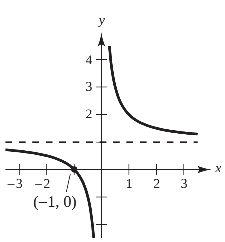

**50.** Intercept: $(-1, 0)$

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 0$

$y' = -\dfrac{1}{x^2}$

No relative extrema




**52.** $f(x) = \dfrac{x - 2}{(x - 1)(x - 3)}$

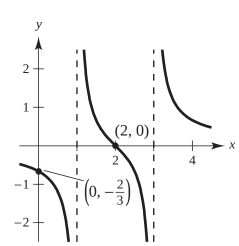
Intercepts: $(2, 0), (0, -2/3)$

Horizontal asymptote: $y = 0$

Vertical asymptotes: $x = 1$ and $x = 3$

$f'(x) = \dfrac{-(x^2 - 4x + 5)}{(x^2 - 4x + 3)^2}$

No relative extrema

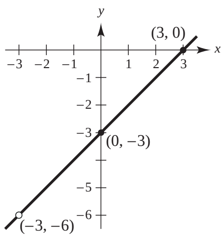


**54.** $g(x) = \dfrac{x^2 - 9}{x + 3} = \begin{cases} x - 3, & x \neq -3 \\ \text{undefined}, & x = -3 \end{cases}$

Intercept: $(3, 0), (0, -3)$

No asymptotes

No relative extrema



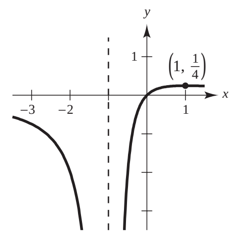

**56.** Intercept: $(0, 0)$

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = -1$

$$y' = \frac{1 - x}{(x + 1)^3}$$

Relative maximum: $\left(1, \frac{1}{4}\right)$




**58.** (a) $\overline{C} = 0.5 + \dfrac{500}{x}$

(b) $\overline{C}(750) = \$7.50$

$\overline{C}(1750) = \$0.90$

(c) $\displaystyle\lim_{x \to \infty} \overline{C} = \lim_{x \to \infty} \left(0.5 + \frac{500}{x}\right) = \$0.50$

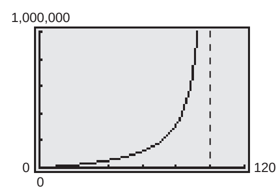
**60.** (a) $C(15) \approx \$14{,}117.65$

$C(50) = \$80{,}000$

$C(90) = \$720{,}000$

(b) $\displaystyle\lim_{x \to 100^-} \frac{80{,}000}{100 - p} = \infty$

(c)

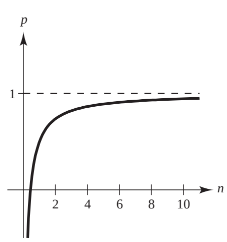



**62.** (a)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $P$ | 0.50 | 0.74 | 0.82 | 0.86 | 0.89 | 0.91 | 0.92 | 0.93 | 0.94 | 0.95 |

(b) $\displaystyle\lim_{n \to \infty} \frac{0.5 + 0.9(n - 1)}{1 + 0.9(n - 1)} = 1$

(c)




**64.** (a) $\overline{P} = \dfrac{R - C}{x} = \dfrac{69.9x - (34.5x + 15{,}000)}{x} = \dfrac{35.4x - 15{,}000}{x} = 35.4 - \dfrac{15{,}000}{x}$

(b) $\overline{P}(1000) = \$20.40$

$\overline{P}(10{,}000) = \$33.90$

$\overline{P}(100{,}000) = \$35.25$

(c) $\displaystyle\lim_{x \to \infty} \left(35.4 - \frac{15{,}000}{x}\right) = \$35.40$

# Section 3.7 Curve Sketching: A Summary

**2.** $y = 2x^2 - 4x + 1$

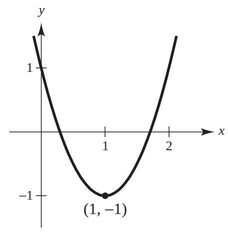
$y' = 4x - 4 = 4(x - 1)$

$y'' = 4$

Intercepts: $(0, 1), \left(1 \pm \sqrt{2}/2, 0\right)$

Relative minimum: $(1, -1)$

Concave upward




**4.** $y = -\frac{1}{3}(x^3 - 3x + 2)$

$y' = -\frac{1}{3}(3x^2 - 3) = 1 - x^2$

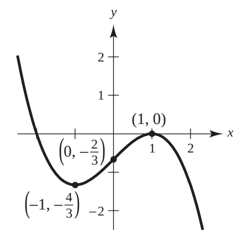
$y'' = -2x$

Intercepts: $\left(0, -\frac{2}{3}\right), (1, 0), (-2, 0)$

Relative minimum: $\left(-1, -\frac{4}{3}\right)$

Relative maximum: $(1, 0)$

Point of inflection: $\left(0, -\frac{2}{3}\right)$




**6.** $y = x^3 + 3x^2 + 3x + 2$

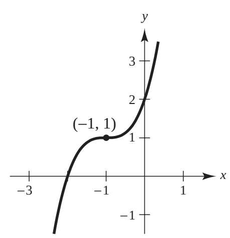
$y' = 3x^2 + 6x + 3 = 3(x + 1)^2$

$y'' = 6x + 6 = 6(x + 1)$

Intercepts: $(0, 2), (-2, 0)$

No relative extrema

Point of inflection: $(-1, 1)$




**8.** $y = -4x^3 + 6x^2$

$y' = -12x^2 + 12x = 12x(1 - x)$

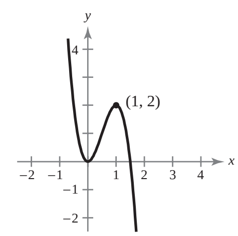
$y'' = -24x + 12 = 12(1 - 2x)$

Critical numbers: $x = 0, 1$

Relative maximum: $(1, 2)$

Relative minimum: $(0, 0)$

Point of inflection: $\left(\frac{1}{2}, 1\right)$




**10.** $y = 3x^4 - 6x^2 = 3x^2(x^2 - 2)$

$y' = 12x^3 - 12x = 12x(x^2 - 1)$

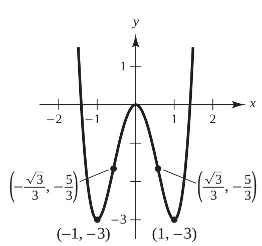
$y'' = 36x^2 - 12 = 12(3x^2 - 1)$

Intercepts: $(0, 0), \left(\pm\sqrt{2}, 0\right)$

Relative maximum: $(0, 0)$

Relative minima: $(\pm 1, -3)$

Points of inflection: $\left(\pm\sqrt{3}/3, -5/3\right)$




**12.** $f(x) = -x^3 + 3x^2 + 9x - 2$

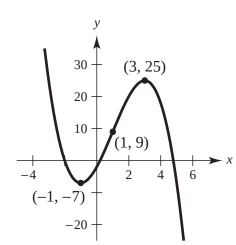
$f'(x) = -3x^2 + 6x + 9 = -3(x + 1)(x - 3)$

$f''(x) = -6x + 6 = -6(x - 1)$

Relative maximum: $(3, 25)$

Relative minimum: $(-1, -7)$

Point of inflection: $(1, 9)$



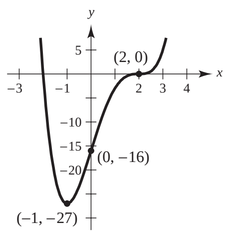
**14.** $f(x) = x^4 - 4x^3 + 16x - 16$

$f'(x) = 4x^3 - 12x^2 + 16 = 4(x + 1)(x - 2)^2$

$f''(x) = 12x^2 - 24x = 12x(x - 2)$

Relative minimum: $(-1, -27)$

Points of inflection: $(0, -16)$, $(2, 0)$




**16.** $f(x) = x^5 + 1$

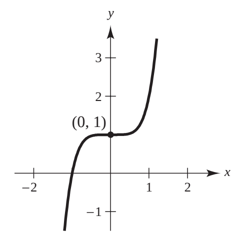
$f'(x) = 5x^4$

$f''(x) = 20x^3$

Intercepts: $(0, 1)$, $(-1, 0)$

No relative extrema

Point of inflection: $(0, 1)$




**18.** $y = (x - 1)^5$

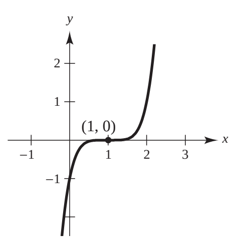
$y' = 5(x - 1)^4$

$y'' = 20(x - 1)^3$

Intercepts: $(0, -1)$, $(1, 0)$

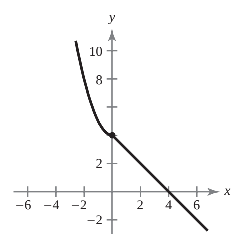
No relative extrema

Point of inflection: $(1, 0)$




**20.** $y = \begin{cases} x^2 + 4, & x < 0 \\ 4 - x, & x \ge 0 \end{cases}$

$y' = \begin{cases} 2x, & x < 0 \\ -1, & x > 0 \end{cases}$



**22.** $y = \dfrac{x}{x^2 + 1}$

$y' = \dfrac{1 - x^2}{(x^2 + 1)^2}$

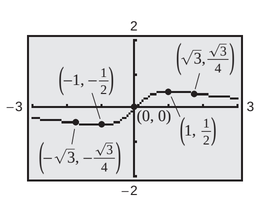
$y'' = \dfrac{2x(x^2 - 3)}{(x^2 + 1)^3}$

Intercept: $(0, 0)$

Relative maximum: $(1, 1/2)$

Relative minimum: $(-1, -1/2)$

Points of inflection: $(0, 0)$, $\left(-\sqrt{3}, -\sqrt{3}/4\right)$, $\left(\sqrt{3}, \sqrt{3}/4\right)$




**24.** $y = 3x^{2/3} - x^2 = x^{2/3}(3 - x^{4/3})$

$y' = 2x^{-1/3} - 2x = \dfrac{2}{\sqrt[3]{x}} - 2x = \dfrac{2(1 - x^{4/3})}{\sqrt[3]{x}}$

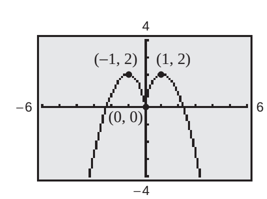
$y'' = -\dfrac{2}{3}x^{-4/3} - 2 = -\dfrac{2}{3x^{4/3}} - 2$

Intercepts: $(0, 0)$, $\left(\pm\sqrt[4]{27}, 0\right)$

Relative maxima: $(-1, 2)$, $(1, 2)$

Relative minimum: $(0, 0)$

Concave downward



**26.** $y = (1 - x)^{2/3}$

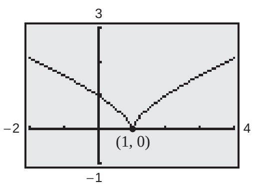
$$y' = -\frac{2}{3}(1 - x)^{-1/3} = -\frac{2}{3\sqrt[3]{1 - x}}$$

$$y'' = -\frac{2}{9}(1 - x)^{-4/3} = -\frac{2}{9(1 - x)^{4/3}}$$

Intercepts: $(0, 1), (1, 0)$

Relative minimum: $(1, 0)$

Concave downward




**28.** $y = x^{-1/3}$

$$y' = -\tfrac{1}{3}x^{-4/3}$$

$$y'' = \tfrac{4}{9}x^{-7/3}$$

No intercepts

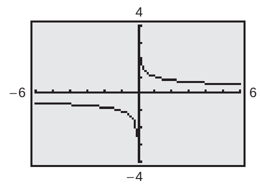
No relative extrema

Decreasing

No points of inflection

Concave downward on $(-\infty, 0)$

Concave upward on $(0, \infty)$




**30.** $y = x^{4/3}$

$$y' = \tfrac{4}{3}x^{1/3}$$

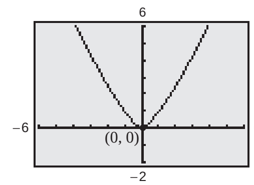
$$y'' = \tfrac{4}{9}x^{-2/3}$$

Intercept: $(0, 0)$

Relative minimum: $(0, 0)$

No points of inflection

Concave upward




**32.** $y = \dfrac{x}{\sqrt{x^2 - 4}}$

Domain: $|x| > 2$

$y' = \dfrac{-4}{(x^2 - 4)^{3/2}}$. Decreasing on $(-\infty, 2)$ and $(2, \infty)$.

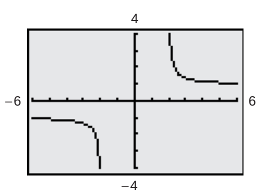
No relative extrema.

$y'' = \dfrac{12x}{(x^2 - 4)^{5/2}}$. No points of inflection.

Vertical asymptote at $x = \pm 2$.

Horizontal asymptote: $y = 1$ for $x > 2$

$y = -1$ for $x < -2$




**34.** $y = \dfrac{x^2 + 1}{x^2 - 2}$

$$y' = -\frac{6x}{(x^2 - 2)^2}$$

$$y'' = \frac{6(3x^2 + 2)}{(x^2 - 2)^3}$$

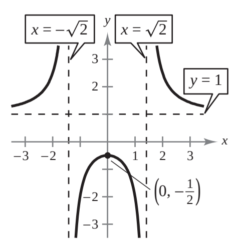
Intercepts: $(0, -1/2)$

Relative maximum: $(0, -1/2)$

Vertical asymptotes: $x = \pm\sqrt{2}$

Horizontal asymptote: $y = 1$

Domain: $(-\infty, -\sqrt{2}), (-\sqrt{2}, \sqrt{2}), (\sqrt{2}, \infty)$




**36.** $y = \dfrac{x^2 - 6x + 12}{x - 4}$

$$y' = \frac{x^2 - 8x + 12}{(x - 4)^2} = \frac{(x - 2)(x - 6)}{(x - 4)^2}$$

$$y'' = \frac{8}{(x - 4)^3}$$

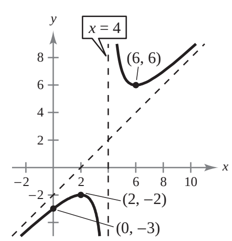
Intercept: $(0, -3)$

Relative maximum: $(6, 6)$

Relative minimum: $(2, -2)$

Vertical asymptote: $x = 4$

Domain: $(-\infty, 4), (4, \infty)$



**38.** $y = x\sqrt{4 - x^2}$

$y' = \dfrac{2(2 - x^2)}{\sqrt{4 - x^2}}$

$y'' = \dfrac{2x(x^2 - 6)}{(4 - x^2)^{3/2}}$

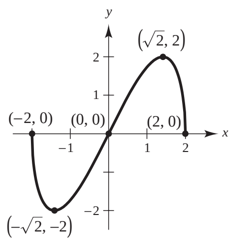
Intercepts: $(0, 0), (2, 0), (-2, 0)$

Relative maximum: $(\sqrt{2}, 2)$

Relative minimum: $(-\sqrt{2}, -2)$

Point of inflection: $(0, 0)$

Domain: $[-2, 2]$




**40.** $y = x + \dfrac{32}{x^2}$

$y' = 1 - \dfrac{64}{x^3} = \dfrac{x^3 - 64}{x^3}$

$y'' = \dfrac{192}{x^4}$

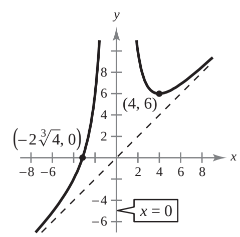
Intercept: $(-2\sqrt[3]{4}, 0)$

Relative minimum: $(4, 6)$

Concave upward

Vertical asymptote: $x = 0$

Domain: $(-\infty, 0), (0, \infty)$




**42.** $y = x^4/(x^4 - 1)$

$y' = -4x^3/(x^4 - 1)^2$

$y'' = 4x^2(5x^4 + 3)/(x^4 - 1)^3$

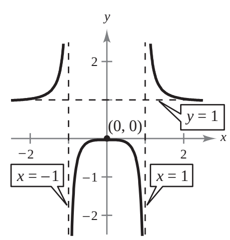
Intercept: $(0, 0)$

Horizontal asymptote: $y = 1$

Vertical asymptotes: $x = \pm 1$

Domain: $(-\infty, -1), (-1, 1), (1, \infty)$

Relative maximum: $(0, 0)$




**44.** Since $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, we have $a > 0$.

$f(x) = x^3 + x^2 - x + 1$

(Solution not unique)

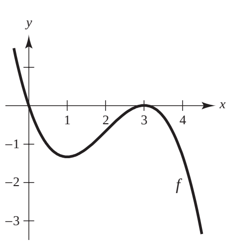
**46.** Since $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = \infty$, we have $a < 0$.

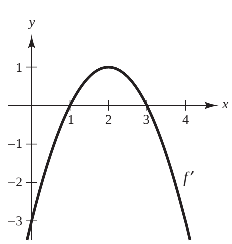
$f(x) = -x^3 + 1$

(Solution not unique)

**48.** $f'(x) = -(x - 1)(x - 3) = -x^2 + 4x - 3$

$f(x) = -\frac{1}{3}x^3 + 2x^2 - 3x$

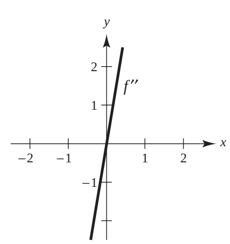


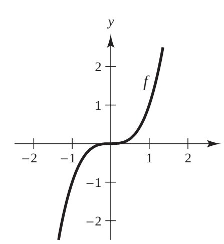




**50.** $f''(x) = 6x$

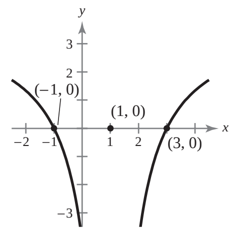
$f' = 3x^2$

$f(x) = x^3$



**52.** $f(x) = \begin{cases} \dfrac{4(x+1)}{x-1}, & x > 1 \\ 0 \quad\quad\quad\;\; , & x > 1 \\ \dfrac{4(x-3)}{x-1}, & x > 1 \end{cases}$



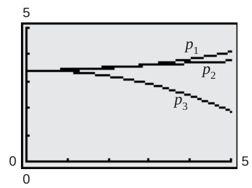

**54.** Model I: $p_1 = 0.03t^2 - 0.01t + 3.39$

Model II: $p_2 = 0.08t + 3.36$

Model III: $p_3 = -0.07t^2 + 0.05t + 3.38$

$t = 0$ corresponds to 2001, $0 \le t \le 5$.

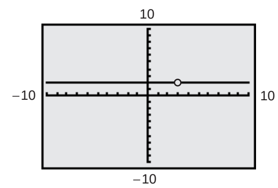
(a)




(b) Models I and II

(c) I most optimistic, III most pessimistic

**56.** $h(x) = \dfrac{6 - 2x}{3 - x} = 2, \quad x \neq 3$




## Section 3.8 Differentials and Marginal Analysis

**2.** $dy = 3x^{1/2}dx = 3\sqrt{x}\,dx$

**4.** $dy = 4(1 - 2x^2)^3(-4x)dx = -16x(1 - 2x^2)^3\,dx$

**6.** $dy = \dfrac{1}{3}(6x^2)^{-2/3}(12x)\,dx = \dfrac{4x}{\sqrt[3]{36x^4}}dx = \dfrac{4}{\sqrt[3]{36x}}\,dx$

**8.** $f(x) = \sqrt{3x}, \quad x = 1, \quad \Delta x = 0.01$

$\Delta y = f(x + \Delta x) - f(x) = \sqrt{3(1.01)} - \sqrt{3(1)} \approx 0.0086387$

**10.** $f(x) = \dfrac{x}{x^2 + 1}, \quad x = 1, \quad \Delta x = 0.01$

$\Delta y = f(x + \Delta x) - f(x) = \dfrac{1.01}{(1.01)^2 + 1} - \dfrac{1}{1^2 + 1} \approx -0.0000247512$

**12.** $y = 1 - 2x^2, \quad x = 0, \quad \Delta x = dx = -0.1$

$dy = -4x\,dx = 0$

$\Delta y = 1 - 2(-0.1)^2 - 1 = -0.02$

**14.** $y = 2x^3 + 1, \quad x = 2, \quad \Delta x = dx = 0.01$

$dy = 6x^2\,dx = 6(2^2)(0.01) = 0.24$

$\Delta y = [2(2.01)^3 + 1] - [2(2)^3 + 1] = 0.241202$

**16.** $dy = 5x^4\,dx$

| $dx = \Delta x$ | $dy$ | $\Delta y$ | $\Delta y - dy$ | $\dfrac{dy}{\Delta y}$ |
|---|---|---|---|---|
| 1.0000 | 80.0000 | 211.0000 | 131.0000 | 0.3791 |
| 0.5000 | 40.0000 | 65.6562 | 25.6562 | 0.6092 |
| 0.1000 | 8.0000 | 8.8410 | 0.8410 | 0.9049 |
| 0.0100 | 0.8000 | 0.8080 | 0.0080 | 0.9901 |
| 0.0010 | 0.0800 | 0.0801 | 0.0001 | 0.9990 |

**18.** $y = \dfrac{1}{x}, x = 2, dy = \dfrac{-1}{x^2}\,dx, \Delta y = \dfrac{1}{x + \Delta x} - \dfrac{1}{x} = \dfrac{-\Delta x}{x(x + \Delta x)}$

| $dx = \Delta x$ | $dy$ | $\Delta y$ | $\Delta y - dy$ | $\dfrac{dy}{\Delta y}$ |
|---|---|---|---|---|
| 1.0 | −0.25 | −0.1667 | 0.0833 | 1.5 |
| 0.5 | −0.125 | −0.1 | 0.025 | 1.25 |
| 0.1 | −0.025 | −0.0238 | 0.00119 | 1.05 |
| 0.01 | −0.0025 | −0.00249 | 0.00001 | 1.005 |
| 0.001 | −0.00025 | −0.00025 | 0.0 | 1.0005 |

**20.** $dy = \dfrac{1}{2\sqrt{x}}\,dx, \quad \Delta y = \sqrt{x + \Delta x} - \sqrt{x}$

| $dx = \Delta x$ | $dy$ | $\Delta y$ | $\Delta y - dy$ | $\dfrac{dy}{\Delta y}$ |
|---|---|---|---|---|
| 1.000 | 0.3536 | 0.3178 | −0.0358 | 1.1126 |
| 0.500 | 0.1768 | 0.1669 | −0.0099 | 1.0590 |
| 0.100 | 0.0354 | 0.0349 | −0.0005 | 1.0123 |
| 0.010 | 0.0035 | 0.0035 | 0.0000 | 1.0012 |
| 0.001 | 0.0004 | 0.0004 | 0.0000 | 1.0001 |

**22.** $f(x) = 3x^2 - 1, \quad (2, 11)$

$f'(x) = 6x, \quad f'(2) = 12$

$y - 11 = 12(x - 2)$

$y = 12x - 13$

$f(2 + 0.01) = 11.1203$

$y(2 + 0.01) = 11.12$

$f(2 - 0.01) = 10.8803$

$y(2 - 0.01) = 10.88$

**24.** $f(x) = \sqrt{25 - x^2}$

$$f'(x) = \frac{1}{2}(25 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{25 - x^2}}$$

$$y - 4 = \frac{-3}{\sqrt{25 - 9}}(x - 3)$$

$$y - 4 = -\frac{3}{4}(x - 3)$$

$$y = -\frac{3}{4}x + \frac{25}{4}$$

$f(3 + 0.01) = 3.99248$

$y(3 + 0.01) = 3.9925$

$f(3 - 0.01) = 4.00748$

$y(3 - 0.01) = 4.0075$

**26.** $N = \dfrac{10(5 + 3t)}{1 + 0.04t}$

$$dN = \frac{(1 + 0.04t)(30) - 10(5 + 3t)(0.04)}{(1 + 0.04t)^2}\,dt = \frac{28}{(1 + 0.04t)^2}\,dt$$

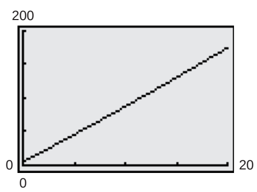
When $t = 5$ and $dt = 6 - 5 = 1$, we have the following.

$$dN = \frac{28}{[1 + 0.04(5)]^2}(1) = \frac{28}{1.44} \approx 19.44$$

The change in herd size will be approximately 19 deer.

**28.** $x = 10,\ dx = \Delta x = 1$

$$\Delta C \approx dC = (0.05x + 8)\,dx$$
$$= (0.05(10) + 8)(1)$$
$$= 8.5 = \$8.50$$




Checking, $C(11) - C(10) = 96.025 - 87.5 = 8.525$

**30.** $x = 15,\ dx = \Delta x = 1$

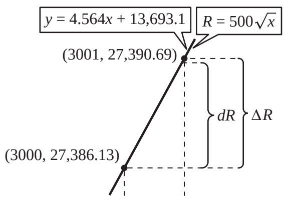
$$\Delta R \approx dR = (50 - 3x)dx$$
$$= (50 - 3(15))(1)$$
$$= 5 \implies \$5.00$$


Checking, $R(16) - R(15) = 416 - 412.5 = 3.8$

**32.** $P = -x^2 + 60x - 100, \quad x = 25, \quad dx = 1$

$$dP = (-2x + 60)\,dx = [-2(25) + 60](1) = \$10$$

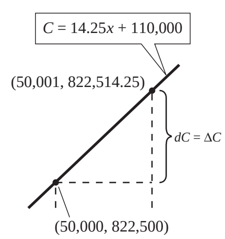
**34.**
$$x = \frac{k}{p^2}$$
$$2500 = \frac{k}{100}$$
$$250{,}000 = k$$
$$x = \frac{250{,}000}{p^2} \implies p^2 = \frac{250{,}000}{x} \implies p = \frac{500}{\sqrt{x}}$$
$$R = xp = x\left(\frac{500}{\sqrt{x}}\right) = 500\sqrt{x}$$
$$dR = \frac{250}{\sqrt{x}}dx = \frac{250}{\sqrt{3000}}(1) \approx \$4.56$$




**36.** $C = 14.25x + 110{,}000$

$$dC = 14.25\,dx = 14.25(1) = \$14.25$$

$\Delta C = dC$ because $C$ is a linear function.



**38.** $A = x^2$

$dA = 2x\,dx$

When $x = 12$ and $dx = \pm 1/64$, we have

$$dA = 2(12)\left(\pm\frac{1}{64}\right) = \pm\frac{3}{8}\text{ in}^2.$$

When $A = 144$, the relative error is

$$\frac{dA}{A} = \frac{\pm 3/8}{144} \approx 0.0026.$$

**40.** $V = x^3, \quad A = 6x^2$

(a) $dV = 3x^2\,dx$

$$\frac{dV}{V} = \frac{3x^2\,dx}{x^3} = \frac{3\,dx}{x}$$

When $x = 12$ and $dx = \pm 0.03$, we have

$$dV = 3(12)^2(\pm 0.03) = \pm 12.96\text{ in}^3.$$

$$\frac{dV}{V} = \frac{3(\pm 0.03)}{12} = \pm 0.0075$$

(b) $dA = 12x\,dx$

$$\frac{dA}{A} = \frac{12x\,dx}{6x^2} = \frac{2\,dx}{x}$$

$$dA = 12(12)(\pm 0.03) = \pm 4.32\text{ in}^2$$

$$\frac{dA}{A} = \frac{2(\pm 0.03)}{12} = \pm 0.005$$

**42.** $C = \dfrac{3t}{27 + t^3}$

$$dC = \frac{3(27 - 2t^3)}{(27 + t^3)^2}\,dt$$

When $t = 1$ and $dt = \frac{1}{2}$, we have

$$dC = \frac{3(25)}{(28)^2}\left(\frac{1}{2}\right) \approx 0.0478.$$

**44.** True. $\Delta y = [a(x + \Delta x) + b] - [ax + b] = a\,\Delta x \Longrightarrow \dfrac{\Delta y}{dx} = a = \dfrac{dy}{dx}$

## Review Exercises for Chapter 3

**2.** $g(x) = (x - 1)^2(x - 3)$

$g'(x) = (x - 1)(3x - 7)$

Critical numbers: $x = 1, x = \frac{7}{3}$

**4.** $f(x) = (x + 1)^3$

$f'(x) = 3(x + 1)^2$

Critical number: $x = -1$

**6.** $g(x) = -x^2 + 7x - 12$

$g'(x) = -2x + 7$

Critical number: $x = \frac{7}{2}$

Increasing on $\left(-\infty, \frac{7}{2}\right)$

Decreasing on $\left(\frac{7}{2}, \infty\right)$

**8.** $f(x) = -x^3 + 6x^2 - 2$

$f'(x) = -3x^2 + 12x = -3x(x - 4)$

Critical numbers: $x = 0, 4$

Increasing on $(0, 4)$

Decreasing on $(-\infty, 0), (4, \infty)$

**10.** (a) $S = 5.8583t^2 - 28.943t^2 - 34.36t + 940.6$, $-2 \le t \le 2$, $t = 0$ corresponds to 2000.

$S' = 17.5749t^2 - 57.886t - 34.36$

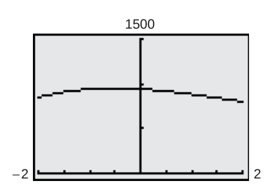
$S$ increasing on $(-2, -0.51)$

(b) $S$ decreasing on $(-0.51, 2)$

(c) Shipments increasing from 1998 to mid-1999.

Shipments decreasing from mid-1999 to 2002.

(d)



**12.** $f(x) = \frac{1}{4}x^4 - 8x$

$f'(x) = x^3 - 8$

Critical number: $x = 2$

| Intervals | $(-\infty, 2)$ | $(2, \infty)$ |
|---|---|---|
| Sign of $f'(x)$ | $-$ | $+$ |
| Conclusion | Decreasing | Increasing |

Relative minimum: $(2, -12)$

**14.** $h(x) = 4 + 10x - x^2$

$h'(x) = 10 - 2x = 2(5 - x)$

Critical number: $x = 5$

Increasing on $(-\infty, 5)$

Decreasing on $(5, \infty)$

Relative maximum: $(5, 29)$

**16.** $s(x) = x^4 - 8x^2 + 3$

$s'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x - 2)(x + 2)$

Critical numbers: $x = 0, 2, -2$

| Interval | $(-\infty, -2)$ | $(-2, 0)$ | $(0, 2)$ | $(2, \infty)$ |
|---|---|---|---|---|
| Sign of $f'$ | $-$ | $+$ | $-$ | $+$ |
| Conclusion | Decreasing | Increasing | Decreasing | Increasing |

Relative minima: $(-2, -13), (2, -13)$

Relative maximum: $(0, 3)$

**18.** $f(x) = \dfrac{2}{x^2 - 1}$

$f'(x) = \dfrac{-4x}{(x^2 - 1)^2}$

Critical number: $x = 0$

Discontinuities: $x = \pm 1$

| Interval | $(-\infty, -1)$ | $(-1, 0)$ | $(0, 1)$ | $(1, \infty)$ |
|---|---|---|---|---|
| Sign of $f'$ | $+$ | $+$ | $-$ | $-$ |
| Conclusion | Increasing | Increasing | Decreasing | Decreasing |

Relative maximum: $(0, -2)$

**20.** $g(x) = x - 6\sqrt{x}, \quad x > 0$

$g'(x) = \dfrac{1 - 3}{\sqrt{x}} = \dfrac{\sqrt{x} - 3}{\sqrt{x}}$

Critical number: $x = 9$

Increasing on $(9, \infty)$

Decreasing on $(0, 9)$

Relative minimum: $(9, -9)$

**22.** $f(x) = x^4 - 2x^3, \quad [0, 2]$

$f'(x) = 4x^3 - 6x^2 = 2x^2(2x - 3)$

Critical numbers: $x = 0, \dfrac{3}{2}$

| $x$ | $f(x)$ | |
|---|---|---|
| 0 | 0 | maximum |
| $\dfrac{3}{2}$ | $-1.6875$ | minimum |
| 2 | 0 | maximum |

**24.** $f(x) = x^3 + 2x^2 - 3x + 4, \quad [-3, 2]$

$f'(x) = 3x^2 + 4x - 3$

Critical numbers: $x = \dfrac{-4 \pm \sqrt{52}}{6} = \dfrac{-2 \pm \sqrt{13}}{3}$

| $x$ | $f(x)$ | |
|---|---|---|
| $-3$ | 4 | |
| $(-2 - \sqrt{13})/3$ | 10.0646 | |
| $(-2 + \sqrt{13})/3$ | 3.1206 | minimum |
| 2 | 14 | maximum |

**26.** $f(x) = 2\sqrt{x} - x, \quad [0, 9]$

$f'(x) = \dfrac{1}{\sqrt{x}} - 1$

Critical number: $x = 1$

| $x$ | $f(x)$ | |
|---|---|---|
| 0 | 0 | |
| 1 | 1 | maximum |
| 9 | $-3$ | minimum |

**28.** $f(x) = -x^4 + x^2 + 2, \quad [0, 2]$

$f'(x) = -4x^3 + 2x = 2x(1 - 2x^2)$

Critical numbers: $x = 0, \dfrac{\sqrt{2}}{2}$

| $x$ | $f(x)$ | |
|---|---|---|
| 0 | 2 | |
| 2 | $-10$ | minimum |
| $\dfrac{\sqrt{2}}{2}$ | $\dfrac{9}{4}$ | maximum |

**30.** $f(x) = \dfrac{8}{x} + x, \quad [1, 4]$

$f'(x) = \dfrac{-8}{x^2} + 1 = \dfrac{x^2 - 8}{x^2}$

Critical number: $x = 2\sqrt{2}$

| $x$ | $f(x)$ | |
|---|---|---|
| 1 | 9 | maximum |
| 4 | 6 | |
| $2\sqrt{2}$ | $4\sqrt{2}$ | minimum |

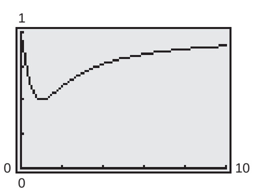
**32.** $O = \dfrac{t^2 - t + 1}{t^2 + 1}, \quad 0 \le t$

The graph of $O$ is:

(a) Lowest level is $\frac{1}{2}$ at $t = 1$.

(b) Highest level is 1 at $t = 0$.

(c) As $t \to \infty$, $O \to 1$.




**34.** $h(x) = x^5 - 10x^2$

$h'(x) = 5x^4 - 20x$

$h''(x) = 20x^3 - 20 = 20(x - 1)(x^2 + x + 1)$

$f''(x)0$ for $x > 1$: concave upward on $(1, \infty)$

$f''(x)0$ for $x < 1$: concave downward on $(-\infty, 1)$

**36.** $h(x) = x^3 - 6x$

$h'(x) = 3x^2 - 6$

$h''(x) = 6x$

$h''(x) > 0$ for $x > 0$: concave upward on $(0, \infty)$

$h''(x) < 0$ for $x < 0$: concave downward on $(-\infty, 0)$

**38.** $f(x) = \dfrac{1}{4}x^4 - 2x^2 - x$

$f'(x) = x^3 - 4x - 1$

$f''(x) = 3x^2 - 4$

$f''(x) = 0$ when $x = \pm\dfrac{2}{\sqrt{3}}$.

Since $f$ changes concavity at $x = \pm 2/\sqrt{3}$, the points of inflection are $\left(-2/\sqrt{3}, (-20/9) + (2/\sqrt{3})\right)$ and $\left(2/\sqrt{3}, (-20/9) - (2/\sqrt{3})\right)$.

**40.** $f(x) = (x + 2)^2(x - 4) = x^3 - 12x - 16$

$f'(x) = 3x^2 - 12$

$f''(x) = 6x = 0$ when $x = 0$.

Since $f$ changes concavity at $x = 0$, $(0, -16)$ is a point of inflection.

**42.** $f(x) = x(x^2 - 3x - 9) = x^3 - 3x^2 - 9x$

$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1)$

$f''(x) = 6x - 6$

Critical numbers: $x = 3, -1$

$f''(3) > 0 \Longrightarrow (3, -27)$ is a relative minimum.

$f''(-1) < 0 \Longrightarrow (-1, 5)$ is a relative maximum.

**44.** $f(x) = (x - 2)^2(x + 2)^2 = x^4 - 8x^2 + 16$

$f'(x) = 4x^3 - 16x = 4x(x^2 - 4)$

$f''(x) = 12x^2 - 16$

Critical numbers: $x = 0, \pm 2$

$f''(0) < 0 \Longrightarrow (0, 16)$ is a relative maximum.

$f''(\pm 2) > 0 \Longrightarrow (2, 0)$ and $(-2, 0)$ are relative minima.

**46.** $R = -\frac{2}{3}(x^3 - 12x^2 - 6), \quad 0 \le x \le 8$

$R' = -\frac{2}{3}(3x^2 - 24x)$

$R'' = -\frac{2}{3}(6x - 24)$

$R'' = 0$ when $x = 4$.

The point of diminishing return is $\left(4, \dfrac{268}{3}\right)$.

**48.** Let $x$ and $y$ be the lengths shown in the figure. By similar triangles we have

$$\frac{5}{x} = \frac{y}{x + 4}.$$

Thus, $y = 5 + (20/x)$. To minimize the length of the hypotenuse, we have the following.

$$z = \sqrt{(x + 4)^2 + y^2} = \sqrt{(x + 4)^2 + \left(5 + \frac{20}{x}\right)^2} = \sqrt{x^2 + 8x + 41 + \frac{200}{x} + \frac{400}{x^2}}$$

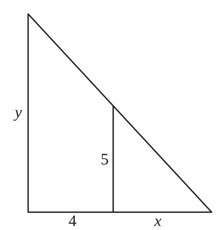
$$\frac{dz}{dx} = \frac{1}{2}\left[(x + 4)^2 + \left(5 + \frac{20}{x}\right)^2\right]^{-1/2}\left(2x + 8 - \frac{200}{x^2} - \frac{800}{x^3}\right)$$

$dz/dx = 0$ when

$$x + 4 - \frac{100}{x^2} - \frac{400}{x^3} = 0$$

$$x^4 + 4x^3 - 100x - 400 = 0$$

$$(x + 4)(x^3 - 100) = 0.$$



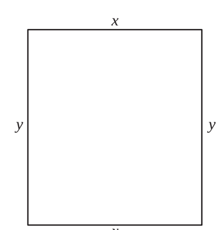

When $x = \sqrt[3]{100}$, $z \approx 12.7$ feet.

**50.** (a) $xy = 4800$

$$\text{Cost} = C = 3(x + 2y) + 4(x) = 7x + 6y = 7x + 6\left(\frac{4800}{x}\right)$$

$$C' = 7 - \frac{28{,}800}{x^2} = 0 \Longrightarrow x \approx 64.14 \text{ feet}$$

$$y \approx 74.83 \text{ feet}$$




(b) If costs increase by \$1, then

$$C = 4(x + 2y) + 5(x) = 9x + 8y = 9x + 8\left(\frac{4800}{x}\right)$$

$$C' = 9 - \frac{38{,}400}{x^2} = 0 \Longrightarrow x \approx 65.32 \text{ feet}$$

$$y \approx 73.48 \text{ feet.}$$

**52.** (a) $N = -2.870t^3 + 79.62t^2 - 639.1t + 3473$, $6 \le t \le 11$, $t = 6$ corresponds to 1996.

$N'(t) = -8.61t^2 + 159.24t - 639.1$

$N''(t) = -17.22t + 159.24$

$N''(t) = 0$ for $t \approx 9.25$

$\frac{dN}{dt}$ increasing on $6 < t < 9.25$, decreasing on $9.25 < t < 11$.

(b) $\lim_{t\to\infty} N = -\infty$ because the cubic term has a negative coefficient.

(c) Answers will vary.

**54.** (a) $P = R - C = xp - C = x(36 - 4x) - (2x^2 + 6)$

$= -6x^2 + 36x - 6$

$P'(x) = -12x + 36 = 0 \Longrightarrow x = 3$ for maximum profit.

(b) $\overline{C} = \frac{C}{x} = \frac{2x^2 + 6}{x} = 2x + \frac{6}{x}$

$\overline{C}' = 2 - \frac{6}{x^2} = 0 \Longrightarrow x^2 = 3 \Longrightarrow x = \sqrt{3} \approx 1.73$ units

**56.** $F = ks^{3/2}$

When $s = 25$, $F = 50$ we have

$$50 = k(25)^{3/2} \Longrightarrow k = \frac{2}{5}$$

$$F = \frac{2}{5}s^{3/2}.$$

The total cost per mile is

$$C = \frac{F}{s} + \frac{100}{s} = \frac{2}{5}s^{1/2} + \frac{100}{s}$$

$$\frac{dC}{ds} = \frac{1}{5}s^{-1/2} - \frac{100}{s^2}.$$

$\frac{dC}{ds} = 0$ when $s = 500^{2/3} \approx 63$ miles per hour.

**58.** $P = xp - C - xt$

$t = \$5$:

$P = x(600 - 3x) - (0.3x^2 + 6x + 600) - 5x$

$= -3.3x^2 + 589x - 600$

$\frac{dP}{dx} = -6.6x + 589$

$\frac{dP}{dx} = 0$ when $x = \frac{589}{6.6}$.

$P\left(\frac{589}{6.6}\right) \approx \$25{,}681.89$

$t = \$10$:

$P = x(600 - 3x) - (0.3x^2 + 6x + 600) - 10x$

$= -3.3x^2 + 584x - 600$

$\frac{dP}{dx} = -6.6x + 584$

$\frac{dP}{dx} = 0$ when $x = \frac{584}{6.6}$.

$P\left(\frac{584}{6.6}\right) \approx \$25{,}237.58$

$t = \$20$:

$P = x(600 - 3x) - (0.3x^2 + 6x + 600) - 20x$

$= -3.3x^2 + 574x - 600$

$\frac{dP}{dx} = -6.6x + 574$

$\frac{dP}{dx} = 0$ when $x = \frac{574}{6.6}$.

$P\left(\frac{574}{6.6}\right) \approx \$24{,}360.30$

**60.** $p = 60 - 0.04x,\ 0 \le x \le 1500,\ \dfrac{dp}{dx} = -0.04$

$$\eta = \frac{p/x}{dp/dx} = \frac{(60 - 0.04)/x}{-0.04} = \frac{x - 1500}{x}$$

$$|\eta| = 1 = \left|\frac{x - 1500}{x}\right| \implies x = |x - 1500| \implies x = 750$$

For $0 < x < 750$, $|\eta| > 1$, elastic.

For $750 < x < 1500$, $|\eta| < 1$, inelastic.

$x = 750$, unit elasticity.

**62.** $p = \sqrt{960 - x},\ 0 \le x \le 960$

$$\frac{dp}{dx} = \frac{-1}{2\sqrt{960 - x}}$$

$$\eta = \frac{p/x}{dp/dx} = \frac{\sqrt{960 - x}/x}{-1/2\sqrt{960 - x}} = \frac{2(x - 960)}{x}$$

$$|\eta| = 1 \implies x = |2x - 1920|$$

$$\implies x = \pm(2x - 1920)$$

$$\implies x = 2x - 1920 \quad \text{or} \quad x = -2x + 1920$$

$$\implies x = 1920 \quad \text{or} \quad x = 640$$

Since $0 \le x \le 960$, select $x = 640$.

Elastic: $(0, 640)$

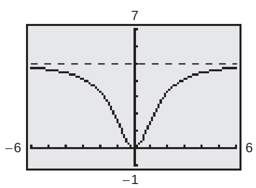
Inelastic: $(640, 960)$

Unit elasticity: $x = 640$

**64.** $$g(x) = \frac{5x^2}{x^2 + 2}$$

$$\lim_{x\to\infty} \frac{5x^2}{x^2 + 2} = 5$$

Horizontal asymptote: $y = 5$
No vertical asymptotes

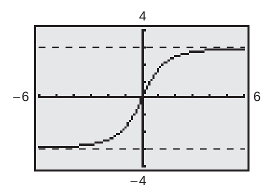



**66.** $$f(x) = \frac{3x}{\sqrt{x^2 + 2}}$$

$$\lim_{x\to\infty} \frac{3x}{\sqrt{x^2 + 2}} = 3$$

$$\lim_{x\to-\infty} \frac{3x}{\sqrt{x^2 + 2}} = -3$$

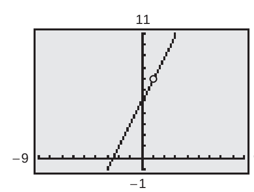
Horizontal asymptotes: $y = \pm 3$
No vertical asymptotes




**68.** $h(x) = \dfrac{2x^2 + 3x - 5}{x - 1} = \dfrac{(x - 1)(2x + 5)}{x - 1} = 2x + 5, \quad x \ne 1$

There are no vertical asymptotes. There is a hole in the graph at $(1, 7)$.
There are no horizontal asymptotes since

$$\lim_{x\to\infty} h(x) = \infty \text{ and } \lim_{x\to-\infty} h(x) = -\infty.$$




**70.** $\displaystyle\lim_{x\to-0^-} \left(3 + \frac{1}{x}\right) = 3 - \infty = -\infty$

**72.** $\displaystyle\lim_{x\to3^-} \frac{3x^2 + 1}{(x - 3)(x + 3)} = -\infty$

**74.** $\displaystyle\lim_{x\to\infty} \frac{3x^2 - 2x + 3}{x + 1} = \lim_{x\to\infty} \frac{3x - 2 + (3/x)}{1 + (1/x)} = \infty$

**76.** $\displaystyle\lim_{x\to-\infty} \left(\frac{x}{x - 2} + \frac{2x}{x + 2}\right) = 1 + 2 = 3$

**78.** $C = 10{,}000 + 48.9x, \quad R = 68.5x$

(a) $\overline{C} = \dfrac{C}{x} = \dfrac{10{,}000}{x} + 48.9$

(b) $\lim\limits_{x\to\infty} \overline{C} = \$48.9$

(c) $\overline{P} = \dfrac{P}{x} = \dfrac{R - C}{x} = 19.6 - \dfrac{10{,}000}{x}$

$\overline{P}(1{,}000{,}000) = \$19.59$

$\overline{P}(2{,}000{,}000) = \$19.595$

$\overline{P}(10{,}000{,}000) = \$19.599$

(d) $\lim\limits_{x\to\infty} \overline{P} = 19.6$

**80.** $f(x) = 4x^3 - x^4$

$f'(x) = 4x^2(3 - x)$

$f''(x) = 12x(2 - x)$

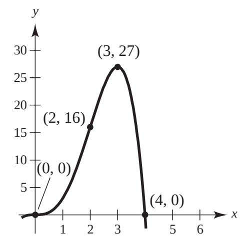
Domain: all real numbers

Range: all real numbers

Intercepts: $(0, 0), (4, 0)$

Relative maximum: $(3, 27)$

Points of inflection: $(0, 0), (2, 16)$




**82.** $f(x) = x^2\sqrt{9 - x^2}$

$f'(x) = \dfrac{3x(6 - x^2)}{\sqrt{9 - x^2}}$

$f''(x) = \dfrac{6x^4 - 81x^2 + 162}{(9 - x^2)^{3/2}}$

Domain: $-3 \le x \le 3$

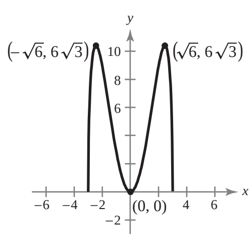
Critical numbers: $x = 0, \ \pm\sqrt{6}$

Relative maxima: $\left(\pm\sqrt{6}, 6\sqrt{3}\right)$

Relative minimum: $(0, 0)$

Inflection points: $(\pm 1.5626, 6.2527)$

No asymptotes.




**84.** $f(x) = \dfrac{2x}{1 + x^2}$

$f'(x) = \dfrac{(1 + x^2)2 - 2x(2x)}{(1 + x^2)^2} = \dfrac{2 - 2x^2}{(1 + x^2)^2}$

$f''(x) = \dfrac{(1 + x^2)^2(-4x) - (2 - 2x^2)2(1 + x^2)2x}{(1 + x^2)^4}$

$= \dfrac{-4x(1 + x^2) - 4x(2 - 2x^2)}{(1 + x^2)^3}$

$= \dfrac{-12x + 4x^3}{(1 + x^2)^3}$

$= \dfrac{4x(x^2 - 3)}{(1 + x^2)^3}$

Domain: all real numbers

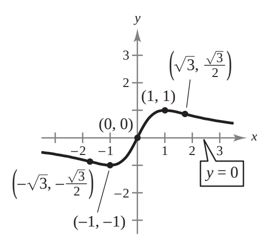
Intercept: $(0, 0)$

Asymptote: $y = 0$

Relative maxima: $(1, 1), (-1, -1)$

Points of inflection: $(0, 0)$,

$$\left(\sqrt{3}, \frac{\sqrt{3}}{2}\right), \left(-\sqrt{3}, -\frac{\sqrt{3}}{2}\right)$$



**86.** $f(x) = x^{4/5}$

$f'(x) = \frac{4}{5}x^{-1/5} = \frac{4}{5\sqrt[5]{x}}$

$f''(x) = -\frac{4}{25}x^{-6/5} = -\frac{4}{25x^{6/5}}$

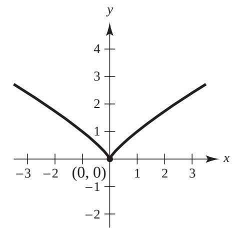
Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

Intercept: $(0, 0)$

Relative minimum: $(0, 0)$

Concave downward on $(-\infty, 0)$ and $(0, \infty)$




**88.** $y = (3x^2 - 2)^3$

$dy = 3(3x^2 - 2)^2(6x)\,dx$

$= 18x(3x^2 - 2)^2\,dx$

**90.** $y = \frac{2 - x}{x + 5}$

$dy = \frac{(x + 5)(-1) - (2 - x)(1)}{(x + 5)^2}\,dx = \frac{-7}{(x + 5)^2}\,dx$

**92.** $C = 1.5\sqrt[3]{x} + 500, \quad x = 125$

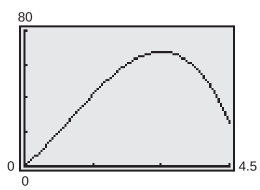
$dC = 0.5x^{-2/3}\,dx = \$0.02$

**94.** $P = 0.003x^2 + 0.019x - 1200, \quad x = 750$

$dP = (0.006x + 0.019)\,dx = \$4.52$

**96.** $E = 22.5t + 7.5t^2 - 2.5t^3, \quad 0 \le t \le 4.5$

(a)




(b) $E'(t) = 0$ for $t = 3$

$E(3) = 67.5$

**98.** $p = 85 - 0.125x$

$\Delta p = [85 - 0.125(8)] - [85 - 0.125(7)] = -\$0.13$

$dp = -0.125\,dx = -\$0.13 \quad (\text{since } dx = \Delta x = 8 - 7 = 1)$