Study Guide # 3

You also need Study Guides # 1 and # 2 for the Final Exam

1. Line integral of a function $f(x,y)$ along C, parameterized by $x = x(t), \quad y = y(t)$ and $a \le t \le b$, is

$$\int_C f(x,y)\; ds = \int_a^b f(x(t),\, y(t))\, \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt\,.$$

(independent of orientation of C, other properties and applications of line integrals of f)

<u>**Remarks**</u>:

(a) $\int_C f(x,y)\, ds$ is sometimes called the *"line integral of f with respect to arc length"*

(b) $\int_C f(x,y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$

(c) $\int_C f(x,y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$

2. Line integral of vector field $\vec{\mathbf{F}}(x,y)$ along C, parameterized by $\vec{\mathbf{r}}(t)$ and $a \le t \le b$, is given by

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt\,.$$

(depends on orientation of C, other properties and applications of line integrals of f)

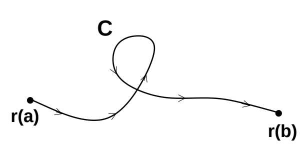
3. Connection between line integral of vector fields and line integral of functions:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C (\vec{\mathbf{F}} \cdot \vec{\mathbf{T}})\, ds$$

where $\vec{\mathbf{T}}$ is the unit tangent vector to the curve C.

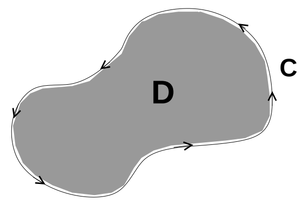
4. If $\vec{\mathbf{F}}(x,y) = P(x,y)\,\vec{\mathbf{i}} + Q(x,y)\,\vec{\mathbf{j}}$, then $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C P(x,y)\, dx + Q(x,y)\, dy$; Work $= \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

5. <u>Fundamental Theorem of Calculus for Line Integrals</u>: $\int_C \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$:



6. A vector field $\vec{\mathbf{F}}(x,y) = P(x,y)\,\vec{\mathbf{i}} + Q(x,y)\,\vec{\mathbf{j}}$ is *conservative* (i.e. $\vec{\mathbf{F}} = \nabla f$) if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$; how to determine a potential function f if $\vec{\mathbf{F}}(\vec{\mathbf{x}}) = \nabla f(\vec{\mathbf{x}})$.

7. <u>Green's Theorem</u>: $\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA$ ($C =$ boundary of D):



As a consequence of Green's Theorem one has

$$\frac{1}{2}\int_C x\,dy - y\,dx = Area(D)$$

8. Del Operator: $\frac{\partial}{\partial x}\,\vec{\mathbf{i}} + \frac{\partial}{\partial y}\,\vec{\mathbf{j}} + \frac{\partial}{\partial z}\,\vec{\mathbf{k}}$; if $\vec{\mathbf{F}}(x,y,z) = P(x,y,z)\,\vec{\mathbf{i}} + Q(x,y,z)\,\vec{\mathbf{j}} + R(x,y,z)\vec{\mathbf{k}}$, then

$$\text{curl } \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad \text{and} \quad \text{div } \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

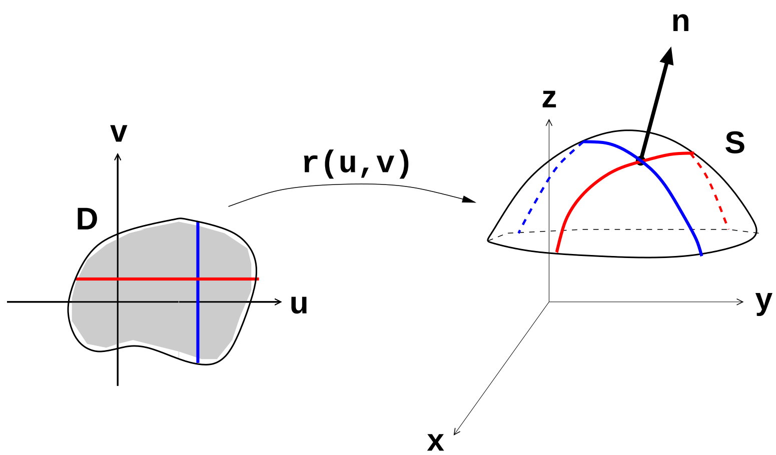
Properties of curl and divergence:

(i) If curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is a conservative vector field (i.e., $\vec{\mathbf{F}}(\vec{\mathbf{x}}) = \nabla f(\vec{\mathbf{x}})$).

(ii) If curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is *irrotational*; if div $\vec{\mathbf{F}} = 0$, then $\vec{\mathbf{F}}$ is *incompressible*.

(iii) *Laplace's Equation*: $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$.

9. Parametric surface S: $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, where $(u,v) \in D$:



Normal vector to surface S : $\vec{\mathbf{n}} = \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$; tangent planes and normal lines to parametric surfaces.

10. Surface area of a surface S:

(i) $A(S) = \iint_D |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\,dA$

(ii) If S is the graph of $z = h(x,y)$ above D, then $A(S) = \iint_D \sqrt{1 + (\partial h/\partial x)^2 + (\partial h/\partial y)^2}\, dA$;

Remark: $dS = |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, dA$ = differential of surface area; while $d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA$

11. The surface integral of $f(x,y,z)$ over the surface S:

(i) $\iint_S f(x,y,z)\, dS = \iint_D f(\vec{\mathbf{r}}(u,v))\, |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, dA.$

(ii) If S is the graph of $z = h(x,y)$ above D, then

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,h(x,y))\, \sqrt{1 + (\partial h/\partial x)^2 + (\partial h/\partial y)^2}\, dA.$$

12. The surface integral of $\vec{\mathbf{F}}$ over the surface S (recall, $d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA$):

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA.$$

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}})\, dS = \iint_D \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA.$$

If S is the graph of $z = h(x,y)$ above D, with $\vec{\mathbf{n}}$ oriented upward, and $\vec{\mathbf{F}} = \langle P, Q, R\rangle$, then

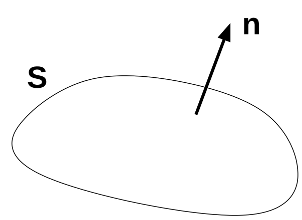
$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \left(-P\frac{\partial h}{\partial x} - Q\frac{\partial h}{\partial y} + R\right) dA.$$

(i) Connection between surface integral of a vector field and a function:

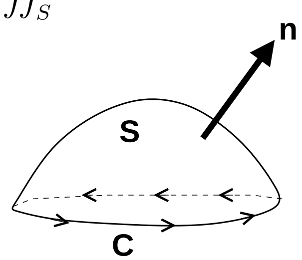
$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}})\, dS.$$

(The above gives another way to compute $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$)

(ii) $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}})\, dS =$ *flux* of $\vec{\mathbf{F}}$ across the surface S.



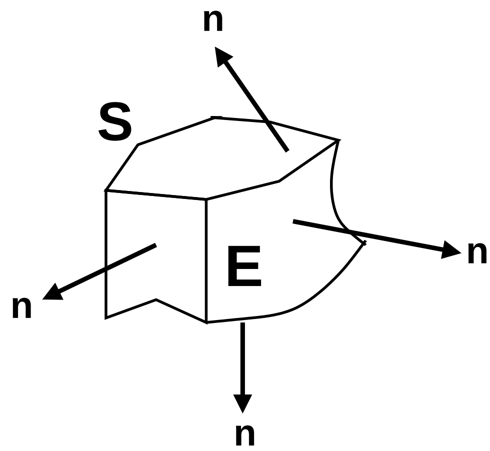
13. Stokes' Theorem: $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S \operatorname{curl} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ (recall, $\operatorname{curl} \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}}$).



$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} =$ *circulation* of $\vec{\mathbf{F}}$ around C.

14. THE DIVERGENCE THEOREM/GAUSS' THEOREM: $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iiint_E \operatorname{div} \vec{\mathbf{F}}\ dV$

(recall, $\operatorname{div} \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}}$).



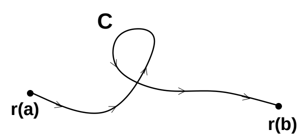
15. Summary of Line Integrals and Surface Integrals:

LINE INTEGRALS	SURFACE INTEGRALS
$C : \vec{\mathbf{r}}(t)$, where $a \le t \le b$	$S : \vec{\mathbf{r}}(u,v)$, where $(u,v) \in D$
$ds = \lvert\vec{\mathbf{r}}\,'(t)\rvert\, dt =$ differential of arc length	$dS = \lvert\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\rvert\, dA =$ differential of surface area
$\int_C ds =$ length of C	$\iint_S dS =$ surface area of S
$\int_C f(x,y,z)\, ds = \int_a^b f(\vec{\mathbf{r}}(t))\, \lvert\vec{\mathbf{r}}\,'(t)\rvert\, dt$ (independent of orientation of C)	$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{\mathbf{r}}(u,v))\, \lvert\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\rvert\, dA$ (independent of normal vector $\vec{\mathbf{n}}$)
$d\vec{\mathbf{r}} = \vec{\mathbf{r}}\,'(t)\, dt$	$d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\ dA$
$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t)\, dt$ (depends on orientation of C)	$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v)) \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\ dA$ (depends on normal vector $\vec{\mathbf{n}}$)
$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \left(\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\right)\ ds$ The *circulation* of $\vec{\mathbf{F}}$ around C	$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S \left(\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\right)\ dS$ The *flux* of $\vec{\mathbf{F}}$ across S in direction $\vec{\mathbf{n}}$

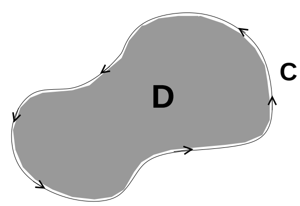
16. Integration Theorems:

<u>Fundamental Theorem of Calculus</u>: $\int_a^b F'(x)\,dx = F(b) - F(a)$

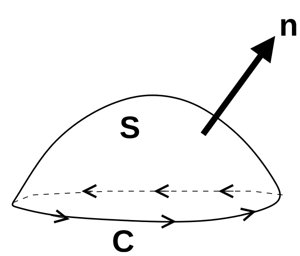
<u>Fundamental Theorem of Calculus For Line Integrals</u>: $\int_a^b \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$



<u>Green's Theorem</u>: $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dA = \int_C P(x,y)\,dx + Q(x,y)\,dy$



<u>Stokes' Theorem</u>: $\iint_S \operatorname{curl} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$



<u>Divergence Theorem</u>: $\iiint_E \operatorname{div} \vec{\mathbf{F}}\ dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$